Approximate Analytical Solutions For Time-Dependent Emden-Fowler-Type Equations By Variational Iteration Method

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Abstract: The main objective of the current study was to solve the time-dependent Emden-Fowler-type equation with singular behavior. The technique used here was called the variational iteration method (VIM). The advantage of this work is twofold. Firstly, the VIM reduces the computational work. Secondly, in comparison with existing techniques, the VIM is an improvement with regard to its accuracy and rapid convergence. The VIM has the advantage of being more concise for analytical and numerical purposes. To show the efficiency of the VIM, several examples are presented. Comparisons with exact solution show that the VIM is a powerful method for the solution of linear and nonlinear equations.

Keywords: Variational iteration method; Emden-Fowler-type equations; Lagrange multiplier.

INTRODUCTION

The variational iteration method (VIM) was first envisioned by He[1] (see also[2–9]). The VIM has successfully been applied to many situations. For example, He[2] solved the classical Blasiu's equation using VIM. He[3] used VIM to give approximate solutions for some well-known non-linear problems. He [4] used VIM to solve utonomous ordinary differential systems. He[5] coupled the iteration method with the perturbation method to solve the well-known Blasius equation. He[7] solved Strongly Nonlinear Equations using VIM. Soliman[10] applied the VIM to solve the KdV-Burger's and Lax's

Seventh-order KdV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa et al.[11]. Momani et al.[12] applied VIM to Helmholtz equation. The VIM has been applied for solving nonlinear differential equations of fractional order by Odibat et al.[13]. Bildik et al.[14] used VIM for solving different types of nonlinear partial differential equations. Wazwaz[15] used VIM to determine rational solutions for the KdV, the K(2,2), the Burgers, and the cubic Boussinesq equations. Wazwaz[16] presented a comparative study between the variational iteration method and Adomian decomposition method. Tamer et al.[17] introduced a modification of VIM. Abbasbandy[18] solved the quadratic Riccati differential equation by He's variational iteration method with considering Adomian's polynomials. Junfeng[19] introduced Variational iteration method to solve two-point boundary value problems.

The heat equation

$$y_{xx} + \frac{r}{x} y_x + af(x,t)g(y) + h(x,t) = y_t,$$

$$0 < x \le L, 0 < t < T, r > 0 \tag{1}$$

models the diffusion of heat perpendicular to surfaces of parallel planes, where $f(x,t)g(y) + h(x,t)$ is the nonlinear heat source, $y(x,t)$ is the temperature, and t is the dimensionless time variable. For $r = 2$, $h(x,t) = 0$, and for steady-state case Eq. (1) is the Emden-Fowler equation[20] given by

$$y'' + \frac{2}{x} y' + af(x)g(y) = 0 \tag{2}$$

where $f(x)$ and $g(x)$ are some given functions of x and y respectively. For $f(x) = 1$ and $g(y) = y^n$, Eq. (2) is the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules[20] and subject to the classical laws of thermodynamics. For other special forms of $g(y)$, the well-known Lane-Emden equation was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents[20,21].

The purpose of this paper is to obtain approximate analytical solutions for the time-dependent Emden-Fowler-type equations, and to do a numerical comparison between VIM and exact solution for solving the time-dependent Emden-Fowler-type equations.

VARIATIONAL ITERATION METHOD

This method, which is a modified general Lagrange's multiplier method[22], has been shown to solve effectively, easily and accurately a large class of nonlinear problems[1-19]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution. To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

$$Lu + Nu = g(x) \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to the VIM[3–9], we can construct a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) +$$
$$\int_0^t \lambda(s)[Lu_n(x,s) + N\tilde{u}(x,s) - g(x,s)]ds, \tag{4}$$

where λ is a general Lagrangian multiplier[22], which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted Variation i.e.
$\delta\tilde{u}_n = 0$.

NUMERICAL RESULTS AND DISCUSSION

We will present numerical and analytical solutions for the time dependent Emden-Fowler type of equations:

$$y_{xx} + \frac{r}{x}y_x + af(x,t)g(y) + h(x,t) = y_t \tag{5}$$

To solve equation (5) by means of He's variational iteration method, we construct a correction functional,

$$y_{n+1}(x,t) = y_n(x,t) +$$
$$\int_0^x \lambda(s)\left[(y_n)_{ss} + \frac{r}{s}(y_n)_s + a\tilde{f}(s,t)\tilde{g}(y_n) + \tilde{h}(s,t) - (\tilde{y}_n)_t\right]ds$$

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) +$$
$$\delta\int_0^x \lambda(s)\left[(y_n)_{ss} + \frac{r}{s}(y_n)_s + a\tilde{f}(s,t)\tilde{g}(y_n) + \tilde{h}(s,t) - (\tilde{y}_n)_t\right]ds$$

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta\int_0^x \lambda(s)\left[(y_n)_{ss} + \frac{r}{s}(y_n)_s\right]ds$$

$$\delta y_{n+1}(x,t) = (1 - \lambda'(x) + \frac{r}{x}\lambda(x))\delta y_n(x,t)$$
$$+\delta\lambda(x)(y_n)_s(x,t) + \int_0^x \delta y_n\left[\lambda''(s) - r\frac{s\lambda' - \lambda(s)}{s^2}\right]ds = 0,$$

where $\tilde{y}_n$ is considered as restricted variations, which mean $\delta\tilde{y}_n = 0$. Its stationary conditions can be obtained as follows

$$1 - \lambda'(x) + \frac{r}{x}\lambda(x) = 0, \ \lambda(x) = 0,$$

$$\lambda''(s) - r\frac{s\lambda'(s) - \lambda(s)}{s^2}\Big|_{s=x} = 0,$$

the Lagrange multipliers, therefore, can be identified as

$$\lambda(s) = \frac{s^r}{(r-1)x^{r-1}} - \frac{s}{r-1},$$

and the following variational iteration formula is obtained

$$y_{n+1}(x,t) = y_n(x,t)$$
$$+\int_0^x \left(\frac{s^r}{(r-1)x^{r-1}} - \frac{s}{r-1}\right)\left[(y_n)_{ss} + \frac{r}{s}(y_n)_s + afg(y_n) + h - (y_n)_t\right]ds \tag{6}$$

Example 1: Consider the homogeneous equation:

$$y_{xx} + \frac{2}{x}y_x - (6 + 4x^2 - \cos t)y = y_t, \tag{7}$$

subject to the initial conditions

$$y(0,t) = e^{\sin t}, \ y_x(0,t) = 0, \tag{8}$$

whose exact solution was found to be[23]:

$$y(x,t) = e^{x^2 + \sin t} \tag{9}$$

To solve equation (8) by means of VIM, we construct a correction functional (see (6)),

$$y_{n+1}(x,t) = y_n(x,t)$$
$$+\int_0^x \left(\frac{s^2}{x} - s\right)\left[(y_n)_{ss} + \frac{2}{s}(y_n)_s - (y_n)_t - (6 + 4s^2 - \cos t)y_n\right]ds \tag{10}$$

For simplicity, we can take an initial approximation $y_0 = y(0,t)$ as given by (11). The next iterate is easily obtained from (7) and is given by:

$$y_1(x,t) = e^{\sin t}\left(1 + \frac{1}{5}x^4 + 5x^2\right) \tag{11}$$

In the same manner, the rest of the components of the iteration formulae (10) can be obtained using the computer program Maple.

Example 2: Consider the inhomogeneous equation:

$$y_{xx} + \frac{2}{x}y_x - (5 + 4x^2)y = y_t + (6 - 5x^2 - 4x^4) \tag{12}$$

with initial conditions

$$y(0,t) = e^t, \ y_x(0,t) = 0, \tag{13}$$

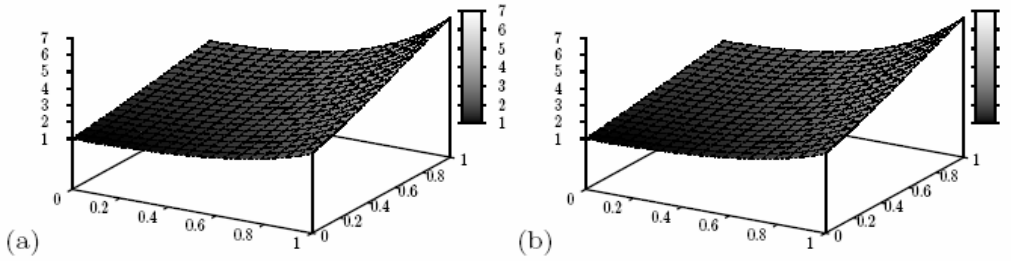and exact solution was found to be[23]:

Fig. 1: Comparison between the (a) exact solution and (b) 5-iterate of VIM for example 1
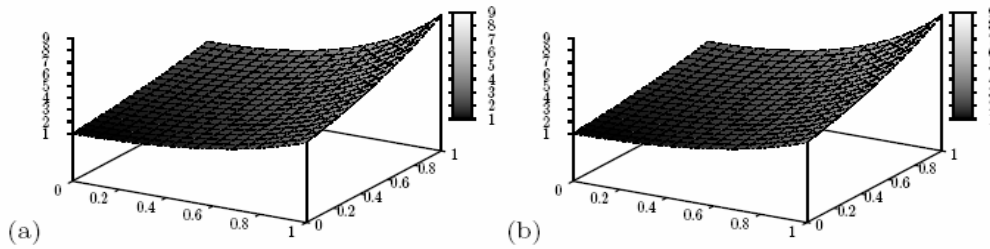

Fig. 2: Comparison between the (a) exact solution and (b) 5-iterate of VIM for example 2
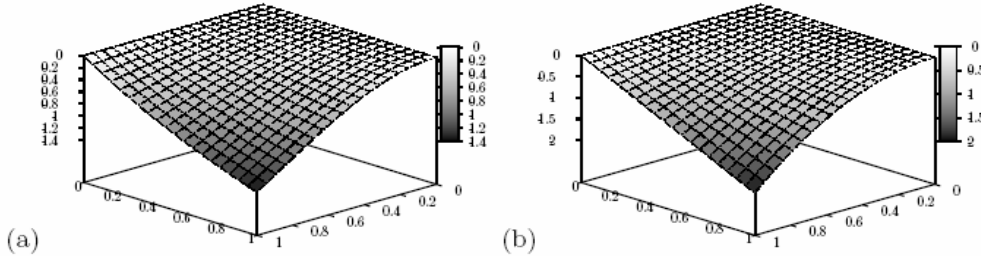

Fig. 3: Comparison between the (a) exact solution and (b) 5-iterate of VIM for example 3

$$y(x,t) = x^2 + e^{x^2+t} \qquad (14)$$

In order to to solve equation (8) by means of VIM, we construct a correction functional (see (6)),

$$y_{n+1}(x,t) = y_n(x,t)$$

$$+ \int_0^x \left(\frac{s^2}{x} - s \right) \left[\begin{array}{l} (y_n)_{ss} + \frac{2}{s}(y_n)_s - (y_n)_t \\ -(5+4s^2)y_n - (6-5s^2-4s^4) \end{array} \right] ds .$$

$$(15)$$

Again, we can take an initial approximation $y_0 = y(0,t)$ as given by (13). The next iterate is easily obtained from (7) and is given by:

$$y_1(x,t) = e^t - \frac{2x^6}{21} + \frac{1}{5}x^4(5-4e^t) + \frac{1}{4}x^4(4e^t-5)$$

$$+ \frac{1}{3}x^2(-6-6e^t) + \frac{1}{2}x^2(6+6e^t)$$

$$(16)$$

Again, the rest of the components of the iteration formulae (15) can be obtained using the computer program Maple.

Example 3.

Consider the following nonlinear time-dependent equation:

$$y_{xx} + \frac{5}{x}y_x - (24t+16t^2x^2)e^y - 2x^2e^{\frac{y}{2}} = y_t, \qquad (17)$$

where initial conditions are

$$y(0,t) = 0, \; y_x(0,t) = 0, \qquad (18)$$

with exact solution[23]:

$$y(x,t) = -2\ln(1+tx^2) \qquad (19)$$

As before the correction functional for Eq. (17) (see (6)),

$$y_{n+1}(x,t) = y_n(x,t)$$

$$+ \int_0^x \left(\frac{s^5}{4x^4} - \frac{s}{4} \right) \left[\begin{array}{c} (y_n)_{ss} + \dfrac{5}{s}(y_n)_s - (y_n)_t \\[2mm] + (24t + 16t^2 s^2)e^{y_n} - 2s^2 e^{\frac{y_n}{2}} \end{array} \right] ds \tag{20}$$

Choosing $y_0 = y(0,t)$ as initial approximation we obtain from (20) an improved approximation:

$$y_1(x,t) = \frac{x^2(16t^2 - 2)}{32} - 2x^2 t + \frac{1}{4} x^4 \left(\frac{1}{2} - 4t^2 \right) \tag{21}$$

As before, the rest of the components of the iteration formulae (15) can be obtained using the computer program Maple.

We now obtain numerical solutions of the time-dependent Emden-Fowler-type equations with singular behaviors. There are some important points to make here. First, the VIM provides the solutions in terms of convergent series with easily computable components. Second, it is clear and remarkable that approximate solutions in all examples using VIM are in good agreement. Third, the VIM technique requires less computational work than existing approaches while supplying quantitatively reliable results. The VIM algorithm is coded in the computer algebra package Maple. The Maple environment variable *Digits* controlling the number of significant digits is set to 20 in all the calculations done in this paper. The results presented in Figs. 1–3 clearly show that good accuracy of VIM.

CONCLUSIONS

This present analysis exhibits the reliable applicability of VIM to solve linear and nonlinear problems with singular feature. In this work we demonstrate that this method can be well suited to attain an analytic solution to the type of examined equations, linear and nonlinear as well. The difficulty in this type of equations, due to the existence of singular point at $x = 0$, is overcome here. Our goal has been achieved by solving the time-dependent Emden-Fowler-type equations and formally obtaining analytical verifiable solutions with a high degree of accuracy without any need to linearization, discretization, or restrictive assumptions. The computational size has been reduced, compared to other existing techniques, and the rapid convergence has been guaranteed through examining several examples. In closing, we point out that other types of ODEs with

singular coefficients, such as Legender's equation, Laguerre's equation, and Bessel's equation, were handled differently, but successively by VIM. VIM justifies its efficiency and presents quite promising results and the method has been gaining its applicability in the literature.

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