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p-Best Approximation on Probabilistic Normed Spaces

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Abstract: We studied the best approximation between two sets in probabilistic normed spaces. We defined the best approximation on these spaces and generalized some definitions such as set of best approximation, p-proximinal set and p-approximately compact relative to any set and proved some theorems about them.

Key words: Probabilistic normed spaces, best approximation, triangle norms

INTRODUCTION

An interesting and important generalization of the notion of metric space was introduced by Menger[1] under the name of statistical metric space, which is now called probabilistic metric space. Menger made a contribution to resolving the interpretative issue of quantum mechanics. The idea of Menger was to use distribution functions instead of nonegative real numbers.

It is also of fundamental importance in probabilistic functional analysis and nonlinear analysis and applications[2,3]. Studies of such spaces by numerous authors followed[4-7]. An important family of probabilistic metric spaces are probabilistic normed spaces (briefly, PN-spaces). Probabilistic normed spaces were introduced by Serstnev in 1963[8].

In this research, we introduce the concept of best approximation in probabilistic normed spaces and present some results. Best approximation play a key role in many areas.

In the sequel after an introduction to probabilistic normed spaces, we define the concept of best approximation in probabilistic normed space and generalize some definitions such as set of best approximation, proximinal set and approximately compact set[9,10].

A distribution function (briefly, a d.f.) is a nondecreasing function F defined on R, with F $(-\infty) = 0$ and F $(-\infty) = 1$. The set of all distribution functions that are left continuous on $(-\infty,\infty)$ will be denoted by Δ. The subset of those d.f's such that F $(0) = 0$ will be denoted by Δ^+ and for every $a \in R$, ε_a is the d.f. defined by,

$$\varepsilon_a = \begin{cases} 0, x \le a \\ 1, x > a. \end{cases}$$

The set Δ, as well as its subsets, can be partially ordered by the usual pointwise order; in this order, ε_0 is the maximal element in Δ^+.

A triangle function is a mapping τ: $\Delta^+\times\Delta^+\to\Delta^+$ that is commutative, associative, nondecreasing in each variable and which has ε_0 as identity.

A continuous t-norm is a continuous binary operation on [0,1] that is commutative, associative, nondecreasing in each variable and has 1 as identity.

Definition 1.1: A probabilistic normed space (briefly denoted by PN space) is a triple (V, υ, τ) where V is a vector space over the field K of real or complex numbers, υ is a function from V into Δ^+, τ is a continuous triangle function and for every choice of x and y in V and any $a \neq 0$ in K the following conditions hold:

- (N1) $\upsilon (x) = \varepsilon_0$ if and only if, $x = \theta$ (θ is the null vector in V)
- (N2) $\upsilon (ax)(t) = \upsilon (x)(t\backslash|a|)$ for all t in R^+,
- (N3) $\upsilon (x+y)\ge\tau(\upsilon(x), \upsilon(y))$

υ is called a probabilistic norm on V (briefly P-norm) and it is called a strong probabilistic norm if for t>0, $x\to\upsilon(x)(t)$ is a continuous map onV.

In the sequel we shall frequently denote the distribution function υ (x) by υ and its value at t by υ_x (t).

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Definition 1.2: Let $G\in\Delta^+$ be different from ε_0 and ε_∞, let (V,||.||) be a normed space and define υ: $V\to\Delta^+$ by $\upsilon_\theta = \varepsilon_\theta$ and, if $x \neq \theta$, by

$$\upsilon_x(t) := G\left(t/\|x\|\right) \quad (t>0)$$

The pair (V, υ) is called the simple space generated by (V||.||) and G.

Definition 1.3: Let $\{P_n\}$ be a sequence in (V, υ, τ) Then

$$p_n \to p \ \text{ iff } \ \upsilon_{p-p_n}(t) > 1-t \ \ \forall t>0$$

Definition 1.4: Let (V, υ, τ) be a PN space. A subset A of X is said to be p-bounded if and only if, there exists t>0 such that $\upsilon_{p-q}(t)>1-t$ for all p, q $\in$ A.

Let (V, υ, τ) be a probabilistic normed space. The open ball $N_p(t)$ with the center $x\in V$ and radius t>0 is defined as follow:

$$N_p(t) = \left\{q : \upsilon_{p-q}(t) > 1-t\right\}$$

Definition 1.5: A probabilistic normed space (V, υ, τ) is said to be a strong probabilistic normed space if for x in V, t>0, $y\to\upsilon_{x-y}(t)$ is a continuous map on V.

MAIN RESULTS

Definition 2.1: Let A and C are two nonempty subset of a probabilistic normed space (V, υ, τ). For t>0, let

$$\begin{aligned}\upsilon_{A-C}(t) &= \sup\left\{\upsilon_{a-c}(t) : (a,c)\in A\times C\right\}\\ &= \sup\left\{\upsilon_{a-C}(t) : a\in A\right\}\end{aligned}$$

An element $a_0 \in A$ is said to be a p-best approximation to C from A if,

$$\upsilon_{a_0-C}(t) = \upsilon_{A-C}(t)$$

Also an element $(a_0, c_0)\in A\times C$ is said to be a p-best approximation pair relative to (A, C) if,

$$\upsilon_{a_0-c_0}(t) = \upsilon_{A-C}(t)$$

We shall denote by $P_A^t(C)$ the set of elements of p-best approximation to C from A i.e.,

$$P_A^t(C) = \left\{a\in A : \upsilon_{a-C}(t) = \upsilon_{A-C}(t), \forall t>0\right\},$$

And dente by $P_A^t(C)$, the set of all elements of p-best approximation pair to (A,C) i.e.,

$$P_{A,C}^t = \left\{(a,c)\in A\times C : \upsilon_{a-c}(t) = \upsilon_{A-C}(t), \forall t>0\right\}$$

Definition 2.2: A sequence converges sub-sequentially if it has a convergent subsequence. In the above notation $x_n \succ x_{n'} \to x_0$ identifies the subsequence and the point to which it converges. Recall that a subset C of an PN space is compact if every sequence in C converges sub-sequentially to an element of C.

Definition 2.3: For a probabilistic normed space X and nonempty subsets A and C a sequence $a_n\in A$ is said to converge in distance to C if $\lim_{n\to\infty}\upsilon_{a_n-C}(t) = \upsilon_{A-C}(t)$.

Example 2.4: Let V = R, (V, ||.||) be a normed space. For $p,q\in V$ define, υ: $V\to\Delta^+$ as $\upsilon_p(t) = \varepsilon_{\|p\|}(t)$. For a, $b\in R^+$, $\tau(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$. Then it is easy to prove that (V, υ, τ) is a PN space. Let A = [0,2], C = [3,4], then for $a\in A$ and $c\in C$, $\varepsilon_{\|3-2\|}(t) > \varepsilon_{\|a-c\|}(t)$. So $\upsilon_{3-2}(t) = \upsilon_{C-A}(t)$ and $\upsilon_{3-A}(t) = \upsilon_{C-A}(t)$. Hence for each t > 0, 3 is a p-best approximation to A from C. Also (3,2) is a p-best approximation pair relative to (A,C).

Definition 2.5: Let (V, υ, τ) be a PN space, the nonempty subset $A\subset V$ is called p-proximinal (p-quasi Chebyshev) set relative to C if $P_A^t(C)$ is nonvoid (compact) for some $C\subset V/A$. Also, for nonempty subsets A and C of B, $A\times C$ is called p-proximinal (p-quasi Chebyshev), pair relative to (A,C) if $P_{A,C}^t$ is nonvoid (compact).

Lemma 2.6: Let A and C be nonempty subsets of a PN space (V, υ, τ) and A is a compact set, then $\bar{A}\cap\bar{C}$ is nonempty if and only if υA-C (t) = 1 for t>0.

Proof: Suppose for all t>0, $\upsilon_{A-C}(t) = 1$. As V is first countable, there exists a sequence $\{a_n\}$ in A such that $\upsilon_{a_n-C}(t)\to\upsilon_{A-C}(t)$. Since A is compact set, there exists a subsequence a_{n_k} and a_0 in A such that $a_{n_k}\to a_0$. Therefore

$$\upsilon_{a_0-C}(t) = \upsilon_{A-C}(t) \ \ \forall t>0$$

Hence $\upsilon_{a_0-C}(t) = 1$, so $a_0\in\bar{C}$.

Conversely, suppose there exists $a\in\bar{A}\cap\bar{C}$, then for t > 0, $\upsilon_{a_0-C}(t) = 1$ and so

$$\upsilon_{A-C}(t) = 1, \ \ \forall t>0$$

Definition 2.7: Let (V, υ, τ) be a PN space and A and C be nonempty subsets of V. We say that the subset A is p- approximately compact relative to C if every sequence $a_n \in A$ with the property that, for all t>0, $\upsilon_{x_n - C}(t) \to \upsilon_{A-C}(t)$ has a subsequence convergent to an element of A.

Theorem 2.8: Let A and C are nonempty subsets of a PN space (V, υ, τ) and A is p- approximately compact relative to C, then A is a p-proximinal set relative to C.

Proof: By definition, there exists $\{a_n\} \subset A$ such that $\upsilon_{a_n - C}(t) \to \upsilon_{A-C}(t)$. Since A is p- approximately compact relative to C, so there exists a subsequence a_{n_k} and $a_0 \in A$ such that $a_{n_k} \to a_0$. Since (V, υ, τ) is a strong PN space, we have, $\upsilon_{a_{n_k} - C}(t) \to \upsilon_{a_0 - C}(t)$. Hence for all t>0, $\upsilon_{a_0 - C}(t) = \upsilon_{A-C}(t)$.

Theorem 2.9: Let A is a p- approximately compact relative to C then, A is p-quasi Chebyshev relative to C.

Proof: Let $\{a_n\}$ be a sequence in $P_A^t(C)$. It is obvious that there exists a subsequence $\{a_{n_k}\}$ and $a_0 \in A$ such that $a_{n_k} \to a_0$ and this complete the proof.

Theorem 2.10: Let A and C be nonempty subsets of a probabilistic normed space (V, υ, τ). If A is p-approximately compact and C is compact, then A is p-approximately compact relative to C.

Proof: Let $a_n \in A$ be any sequence converging in distance to C and let the sequence $c_n \in C$ for all t>0 satisfies, $\lim \upsilon_{a_n - c_n}(t) = \upsilon_{A-C}(t)$. Since C is compact, $c_n \succ c_{n'} \to c_0 \in C$.

Hence

$$\upsilon_{A-C}(t) \geq \upsilon_{a_{n'} - c_0}(t) \geq \tau(\upsilon_{a_{n'} - c_{n'}}, \upsilon_{c_{n'} - c_0})$$
$$(t) \geq \tau(\upsilon_{A-C}, \varepsilon_0)(t)$$

Then $a_{n'}$ converges in distance to c and, since $\upsilon_{A-C}(t) = \upsilon_{A-c_0}(t)$ and A is approximately compact, $a_n \succ a_{n'} \to a_0 \in A$; that is, a_n converges subsequentially to an element of A.

Theorem 2.11: Let A and C be nonempty subsets of a probabilistic normed space (V, υ, τ). If A is p-approximately compact and C is compact, then A×C is p-quasi Chebyshev set relative to (A,C).

Proof: Let $(a_n, c_n) \in A\times C$ be any sequence in $P^t_{A,C}$ Then for every t>0, $\upsilon_{a_n - c_n}(t) \to \upsilon_{A-C}(t)$. Since C is compact, $c_n \succ c_{n'} \to c_0 \in C$. Hence,

$$\upsilon_{A-c_0}(t) \geq \upsilon_{a_{n'} - c_0}(t) \geq \tau(\upsilon_{a_{n'} - c_{n'}}, \upsilon_{c_{n'} - c_0})(t)$$
$$= \tau(\upsilon_{A-C}, \varepsilon_0)(t)$$
$$\geq \tau(\upsilon_{A-c_0}, \varepsilon_0)(t)$$

Therefore $\lim \upsilon_{a_{n'} - c_0}(t) = \upsilon_{A-c_0}(t)$ since A is p-approximately compact, $a_n \succ a_{n'} \to a_0 \in A$. Hence,

$$\upsilon_{a_0 - c_0}(t) = \upsilon_{A-C}(t)$$

Lemma 2.12: Let A and C be nonempty subsets of a probabilistic normed space (V, υ, τ). If A is p-approximately compact and p-bounded and C is p-boundedly compact, then A is p-approximately compact to C.

Proof: Let a_n be any sequence converges in distance to C and let $c_n \in C$ satisfies

$$\upsilon_{a_n - c_n}(t) \to \upsilon_{A-C}(t)$$

As a_n is p-bounded, so is c_n. Since C is p-boundedly compact, $c_n \succ c_{n'} \to c_0 \in V$.

Lemma 2.13: Let A and C be nonempty subsets of a probabilistic normed space (V, υ, τ). If A is closed and p-boundedly compact and C is p-bounded, then A is approximately compact relative to C.

Proof: Suppose a_n be a sequence such that $\upsilon_{a_n - C}(t) \to \upsilon_{A-C}(t)$ and choose c_n in C such that

$$\upsilon_{a_n - c_n}(t) \to \upsilon_{A-C}(t)$$

As c_n is p-bounded, so is a_n; hence, $a_n \succ a_{n'} \to a_0 \in A$, which complete, the proof.

Theorem 2.14: Let A and C be nonempty subsets of a probabilistic normed space (V, υ, τ). If A is p-proximinal and C is compact, then A×C is p-proximinal pair relative to (A, C).

Proof: Suppose $c_n \in C$ satisfies $\lim \upsilon_{c_n - A}(t) = \upsilon_{A-C}(t)$. By compactness of C, $c_n \succ c_{n'} \to c_0 \in C$ so $\upsilon_{A-c_0}(t) = \upsilon_{A-C}(t)$. Since A is proximinal, there exists

$a_0 \in A$ such that for all t>0, $\upsilon_{a_0-c_0}(t) = \upsilon_{A-c_0}(t)$, so $\upsilon_{a_n-c_n}(t) = \upsilon_{A-C}(t)$.

Theorem 2.15: Let A and C be nonempty subsets of a probabilistic normed space (V, υ, τ). If A is p-proximinal and p-bounded and C is closed and p-boudedly compact, then A×C is p-proximinal pair relative to (A,C).

Proof: Suppose $c_n \in C$ satisfies $\lim \upsilon_{c_n-A} = \upsilon_{A-C}$. Since A is p-bounded, c_n must also be p-bounded so $c_n \succ c_{n'} \to c_0 \in C$.

Lemma 2.16: Let (V, υ, τ) and (W, υ, τ) be PN spaces. If we define

$$\upsilon_{(x,y)-(x',y')} = \tau\left(\upsilon_{x-x'}, \upsilon_{y-y'}\right)$$

then (V, W, υ, τ) is a probabilistic normed space and the topology induced on V×W is the product topology.

Theorem 2.17: Let S and P be nonempty subsets of PN spaces (V, υ, τ) and (W, υ, τ), respectively and P is compact. If S is p-boundedly compact or approximately compact, then so is S×P.

Proof: Let S be boundedly compact, we show that any p-bounded sequence (s_n, p_n) in S×P has a convergent subsequence. Indeed, by definition of the product probabilistic norm, s_n is p-bounded and since S is p-boundedly compact, $s_n \succ s_{n'} \to s_0 \in S$. By compactness of P, $p_n \succ p_{n'} \to p_0 \in P$. Hence,

$$(s_n, p_n) \succ (s_{n''}, p_{n''}) \to (s_0, p_0) \in V \times W.$$

Now, let S be approximately compact, (x,y) be any element in V×W and suppose that (s_n, p_n) is a sequence in S×P which converges in distance to (x,y), that is,

$$\lim \upsilon_{(s_n,p_n)-(x,y)}(t) = \upsilon_{(x,y)-S\times P}(t)$$

By compactness of P, $p_n \succ p_{n'} \to p_0 \in P$. Hence,

$$\lim_{n'\to\infty} \upsilon_{(s_{n'},p_0)-(x,y)}(t) = \upsilon_{(x,y)-S\times P}(t)$$

So,

$$\lim_{n'\to\infty} \tau\left(\upsilon_{s_{n'}-x}, \upsilon_{p_o-y}\right)(t) = \tau\left(\upsilon_{x-S}, \upsilon_{y-P}\right)(t)$$

Since

$$\upsilon_{p_o-y}(t) \le \upsilon_{y-P}(t),$$

then $\lim_{n'\to\infty} \upsilon_{s_{n'}-x}(t) \ge \upsilon_{x-S}(t)$, which implies $\lim_{n'\to\infty} \upsilon_{s_{n'}-x}(t) = \upsilon_{x-S}(t)$. Hence, s'_n converges in distance to x and since S is approximately compact, $s_{n'} \succ s_{n''} \to s_0 \in S$. Therefore, $(s_n, p_n) \succ (s_{n''}, p_{n''}) \to (s_0, p_0) \in S \times P$, i.e. S×P is approximately compact.

Theorem 2.18: Let A and C be nonempty subsets of a PN space (V, υ, τ). If A is approximately compact and C is compact, then $K = \left\{a \in A : \exists c \in C, \upsilon_{a-c}(t) = \upsilon_{A-c}(t)\right\}$ is compact.

Proof: Let a_n be a sequence in K and for every $n \in N$ choose c_n in C such that a_n minimizes the distance from A to c_n. Since C is compact, $c_n \succ c_{n'} \to c_0 \in C$. Hence,

$$\begin{aligned}
\upsilon_{c_0-A}(t) \ge \upsilon_{a_{n'}-c_0}(t) &\ge \tau\left(\upsilon_{a_{n'}-c_{n'}}, \upsilon_{c_{n'}-c_0}\right)(t) \\
&= \tau\left(\upsilon_{A-c_{n'}}, \upsilon_{c_{n'}-c_0}\right)(t) \\
&\ge \tau\left(\tau\left(\upsilon_{A-c_0}, \upsilon_{c_0-c_{n'}}\right), \upsilon_{c_{n'}-c_o}\right)(t) \\
&= \tau\left(\tau\left(\upsilon_{A-c_0}, \varepsilon_o\right), \varepsilon_o\right)(t)
\end{aligned}$$

Then $\lim \upsilon_{a_{n'}-c_0}(t) = \upsilon_{c_o-A}(t)$. Therefore $a_{n'}$ converges in distance to c_o. Since A is approximately compact, so it converges subsequentially.

CONCLUSION

In this research we introduce the concept of best approximation in probabilistic normed spaces and present some results.

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