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A Fixed Point Theorems in L-Fuzzy Quasi-Metric Spaces

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Abstract: At first we considered the L-fuzzy metric space notation which is useful in modeling some phenomena where it is necessary to study the relationship between two probability functions as well observed in Gregori *et al*. [A note on intuitionistic fuzzy metric spaces. Chaos, Solitons and Fractals 2006; 28: 902-905]. Then we introduced the concept of fixed point theorem in L-fuzzy metric space and finally, showed that every contractive mapping on an L-fuzzy metric space has a unique fixed point.

Key words: Fixed-point theorem, fuzzy sets

INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh[1] in 1965. Various concepts of fuzzy metric spaces were considered in George and Veeramani[2] and Mihet[3,4].

In this research, at first we shall adopt the usual terminology, notation and conventions of L-fuzzy metric spaces introduced by Saadati *et al*.[5] which are a generalization of fuzzy metric spaces[2] and intuitionistic fuzzy metric spaces[6,7]. Then we consider the fixed point theorem on such spaces and show that every contractive mapping on non-Archimedean L-fuzzy metric space has a unique fixed point.

Definitions 1.1: Goguen[8] let $L = (L, \leq_L)$ be a complete lattice and U a non-empty set called universe. An L-fuzzy set A on U is defined as a mapping. $A: U \to L$. For each u in U, A(u) represents the degree (in L) to which u satisfies A.

Classically, a triangular norm T on $([0,1],\leq)$ is defined as an increasing, commutative, associative mapping $T: [0,1]^2 \to [0,1]$ satisfying $(1, x) = x$ for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $L = (L, \leq_L)$.

Definitions 1.2: A triangular norm (t-norm) on L is a mapping $\tau: L^2 \to L$ satisfying the following conditions:

- $(\forall x \in L)(\tau(x, 1_L) = x)$ (boundary condition)
- $(\forall (x,y) \in L^2)(\tau(x,y) = \tau(y,x))$ (commutativity)
- $(\forall (x,y,z) \in L^3)(\tau(x,\tau(y,z)) = \tau(\tau(x,y),z))$ (associativity)
- $(\forall (x,x',y,y') \in L^4)$ $(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow \tau(x,y) \leq_L \tau(x',y'))$ (monotonicity)

The t-norm τ is Hadzic type if $\tau(x,y) \geq_L \wedge(x,y)$ for every $x, y \in L$ where

$$\wedge(x,y) = \begin{cases} x, \text{if } x \leq_L y, \\ y, \text{if } y \leq_L x. \end{cases}$$

Triangle norms are recursively defined by $\tau^1 = \tau$ and

$$\tau^n\left(x_{(1)},\dots,x_{(n+1)}\right) = \tau\left(\tau^{n-1}\left(x_{(1)},\dots,x_{(n)}\right), x_{(n+1)}\right)$$

for $n \geq 2$, $x_{(i)} \in L$ and $i \in \{1,2,\dots,n+1\}$.

Definition 1.3: Deschrijver *et al*.[9] A negator on L is any decreasing mapping $N: L \to L$ satisfying $N(0_L) = 1_L$ and $N(1_L) = 0_L$. If $N(N(x)) = x$ for all $x \in L$, then N is called an involutive negator.

In this research the negator $N: L \to L$ is fixed. The negator N_s on $([0,1], \leq)$ defined as $N_s(x) = 1-x$, for all $x \in [0,1]$, is called the standard negator on $([0,1], \leq)$.

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Definition 1.4: The triple (X, M, τ) is said to be an L-fuzzy quasi-metric space if X is an arbitrary (non-empty) set, τ is a continuous t-norm on L and *M* is an L-fuzzy set on $X^2\times]0,+\infty[$ satisfying the following conditions for every x, y, z in X and t, s in $]0,+\infty[$:

- $M(x,y,t) >_L 0_L$
- $M(x,y,t) = M(y,x,t) = 1_L$ for all $t > 0$ if and only if $x = y$
- $\tau(M(x,y,t), M(y,z,s)) \leq_L M(x,z,t+s)$
- $M(x,y,.):]0,\infty[\to L$ is continuous
- $\lim_{t\to\infty} M(x,y,t) = 1_L$.

In this case, M is called an L-fuzzy quasi-metric.

If, in the above definition, the triangular inequality (c) is replaced by

$$\text{(NA) } \tau(M(x,y,t), M(y,z,s)) \leq_L M(x,z,\max\{t,s\})\ \forall x,y,z \in X,\quad \forall t,s > 0$$

or, equivalently,

$$\tau(M(x,y,t), M(y,z,t)) \leq_L M(x,z,t)\quad \forall x,y,z \in X,\quad t > 0.$$

Then the triple (X, M, τ) is called a non-Archimedean L-fuzzy quasi-metric space[3,4].

For $t \in\ ]0,+\infty[$, we define the closed ball B [x, r, t] with center $x \in X$ and radius $r \in L\backslash\{0_L, 1_L\}$, as

$$B[x,r,t] = \{y \in X : M(x,y,t) \geq_L N(r)\}.$$

Definition 1.5: A sequence $\{x_n\}_{n\in N}$ in an L-fuzzy quasi-metric space (X, M, τ) is called a right (left) Cauchy sequence if, for each $\varepsilon \in L\backslash\{0_L\}$ and $t > 0$, there exists $n_0 \in N$ Such that, for all $m \geq n \geq n_0$ ($n \geq m \geq n_0$),

$$M(x_m, x_n, t) >_L N(\varepsilon).$$

The sequence $\{x_n\}_{n\in N}$ is said to be convergent to $x \in X$ in the L-fuzzy quasi-metric space (X, M, τ) (denoted by $x_n \xrightarrow{M} x$) if $M(x_n,x,t) = M(x,x_n,t) \to 1_L$, whenever $n\to+\infty$ for every $t>0$. An L-fuzzy quasi-metric space is said to be right (left) complete if and only if every right (left) Cauchy sequence is convergent.

Definition 1.6: Let (X, M, τ), be an L-fuzzy metric space and let N, be a negator on L. Let *A* be a subset of X, then the LF-diameter of the set A is the function defined as:

$$\delta_A(s) = \sup_{t<s} \inf_{x,y\in A} M(x,y,t).$$

A sequence $\{A_n\}_{n\in N}$ of subsets of an L-fuzzy quasi-metric space is called decreasing sequence if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$.

The following lemma gives conditions under which the intersection of such sequences is nonempty.

Lemma 1.7: Let (X, M, τ) be a left complete L-fuzzy metric space and let $\{A_n\}_{n\in N}$, be a decreasing sequence of nonempty closed subsets of X such that $\delta_{An}(t)\to 1_L$ as $n\to\infty$. Then $A = \bigcap_{n=1}^{\infty} A_n$ contains exactly one point.

Proof: From the assumption $\delta_{An}(t)\to 1_L$, it is evident that the set A can`t contain more than one element. So it is enough to show that A is nonempty. Let x_n be a point in A_n. Since $\delta_{An}(t)\to 1_L$, by definition of 5F-diameter, $\{x_n\}_{n\in N}$, is a left Cauchy sequence in X. Since (X, M, τ), is left complete, $\{x_n\}_{n\in N}$, has a limit x. We show that x is in A and for this it suffices to show that x is in A_{n_0}, for a fixed but arbitrary n_0. If $\{x_n\}_{n\in N}$, has only finitely many distinct points, then Ax is that point infinitely repeated and is therefore in A_{n_0}. If $\{x_n\}_{n\in N}$ has infinitely many distinct points, then x is a limit point of the set of points of the sequence, so it is a limit point of the subset $\{x_n : n \geq n_0\}$ of the set of the points of the sequence which implies it is a limit point of A_{n_0} and since A_{n_0} is closed, it is in A_{n_0}.

Corollary 1.8: Let (X, M, τ) be a left complete L-fuzzy metric space and let $\{A_i\}_{i\in I}$ be a family of closed subsets of X, which has the finite intersection property and for each $\varepsilon>0$, contains a set of LF-diameter less than ε, then $\bigcap_{i\in I} A_i \neq \phi$.

Proof: For each n= 1,2,… let $i_n \in I$ denote an index such that

$$\delta_{A_{i_n}}(t) = M\left(x, y, \frac{1}{n}\right)$$

for every $x \neq y$. The set $A_n = \bigcap_{j\leq n} A_{i_j}$ satisfy the assumption of the last lemma. Therefore $\bigcap_{n=1}^{\infty} A_n$, contains exactly one point say x_0. Then $x_0 \in A_{i_1}$, for $i_1 \in I$. Indeed define $A'_n = A_{i_1} \cap A_n$ for n = 1,2,…. Now

$$\phi \neq \bigcap_{n=1}^{\infty} A'_n = A_{i_1} \cap \left(\bigcap_{n=1}^{\infty} A_n\right) = A_{i_1} \cap \{x_0\}.$$

Definition 1.9: Let (X, M, τ) be an L-fuzzy metric space. A mapping Δ: X→X is said to be contractive if whenever x and y are distinct point in X, we have

$$M(\Lambda x, \Lambda y, t) >_L M(x, y, t).$$

MAIN RESULT

Theorem 2.1: Let (X, M, τ) be non-Archimedean L-fuzzy metric space, in which τ is Hadzic type. If Δ: X→X is a contractive mapping then Δ has a unique fixed point.

Proof: Let $B_x = B[x, \eta, t]$ with $\eta(x, t) = N(m(x, \Delta x, t))$ and t > 0. Let A be the collection of all these balls for all $x \in X$. The relation $B_x \leq B_y$ if and only if $B_y \subseteq B_x$ is a partial order in A. Consider a totally ordered subfamily A_i of A. From Corollary 1.8, we have,

$$\bigcap_{B_x \in A_1} B_x = B \neq \varphi.$$

Let $y \in B$ and $B_x \in A_1$, then

$$M(x, y, t) \geq_L N(N(M(x, \Lambda x, t))) = M(x, \Lambda x, t) \quad (1)$$

Now, if $x_0 \in B_y$, then

$$\begin{aligned} M(x_0, y, t) &\geq_L N(N(M(y, \Lambda y, t))) \\ &\geq_L \tau^2(M(y, x, t), M(x, \Lambda x, t), M(\Lambda x, \Lambda y, t)) \\ &\geq_L M(x, \Lambda x, t). \end{aligned}$$

Thus

$$M(x_0, y, t) \geq_L M(x, \Lambda x, t) \quad (2)$$

Now, by using (1) and (2) we obtain

$$\begin{aligned} M(x_0, x, t) &\geq_L \tau(M(x_0, y, t), M(x, y, t)) \\ &\geq_L \tau(M(x, \Lambda x, t), M(\Lambda x, x, t)) \\ &\geq_L M(x, \Lambda x, t). \end{aligned}$$

Therefore $x_0 \in B_x$ and $B_y \subseteq B_x$ implies that $B_x \leq B_y$ for all $B_x \in A_1$. Thus B_y is an upper bound in A for family A_1. Hence by Zorn's Lemma, A has a maximal element, say, B_z, for some $z \in X$. We claim that z = Δz.

Suppose that z ≠ Δz. Since Δ is contractive, therefore

$$M(\Lambda z, \Lambda^2 z, t) >_L M(z, \Lambda z, t),$$

where $\Delta^2 = \Delta 0 \Delta$ and

$$\Lambda z \in B[\Lambda z, \eta(\Lambda z, t), t] \cap B[z, \eta(z, t), t]$$

Therefore $B_{\Delta z} \subseteq B_z$ and z is not in $B_{\Delta z}$. Thus $B_{\Delta z} \subset B_z$, which contradicts the maximality of B_z. Hence Δ has a fixed point.

Uniqueness easily follows from contractive condition.

CONCLUSION

In this research we introduce the concept of fixed point theorem in L-fuzzy metric spaces and present some results.

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