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Topological Conjugacy Between Seizure and Flat Electroencephalography

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Abstract: Problem statement: Seizure and Flat EEG which are modeled as two distinct dynamical systems share the same dynamics in the topological viewpoint. **Approach:** Motions of seizure and Flat EEG of the dynamical systems were written as set points. A function that maps between these two sets is then built. By using topological conjugacy, they are shown to share the same dynamics. **Results:** Seizure and Flat EEG were shown to share the same dynamics along with other properties such as order isomorphic, homeomorphic and the unique representation of any event during seizure as an open set. **Conclusion:** This study shows that the dynamics of seizure can be transported to Flat EEG.

Key words: Dynamical system, EEG, Fuzzy C-Means (FCM), order isomorphism, topological conjugacy, dynamic functions, temporal ordering, order isomorphic, dynamic trajectory, dimensional space, FTTM

INTRODUCTION

Epilepsy is a general term used for a group of disorders that cause disturbances in electrical signal of the brain. In epilepsy there is a miniature brainstorm of certain groups of brain cells and this is often associated with a sudden and involuntary contraction of a group of muscles and loss of consciousness. It can happen in a small area of the brain or the whole brain. Depending on the part of the brain that is affected, it causes involuntary changes in body movement or function, sensation, awareness, or behavior where these changes are known as epileptic seizure.

Electroencephalography (EEG) is the recording of electrical activity originating from the brain. It plays an important diagnostic role in epilepsy and provides supporting evidence of a seizure disorder as well as assisting with classification of seizures. EEG has been used extensively to record the abnormal brain activity associated with epileptic seizures. It is recorded on the surface of the scalp using electrodes, thus the signal is retrievable non-invasively. The type of activity and the area of the brain that is recorded from EEG will assist the physician in prescribing the correct medication for certain type of epilepsy. Patients with epilepsy that cannot be controlled by medication will often have surgery in order to remove the damaged tissue. Thus the EEG plays an important role in localizing this tissue.

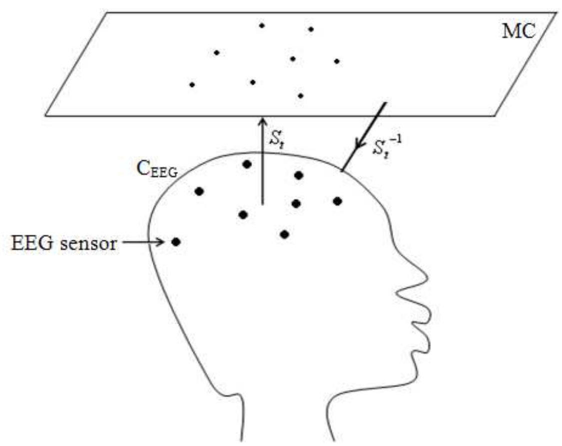

Fig 1: EEG projection

Literature review: Fuzzy Topographic Topological Mapping (FTTM) is a novel model for solving neuromagnetic inverse problem (Ahmad *et al*., 2008). The model is consists of four elements, Magnetic Contour Plane (MC), Base Magnetic Place (BM), Fuzzy Magnetic Field (TM) and Topographic Magnetic Field (TM) each homeomorphic to each other (Ahmad *et al*., 2005). The novel model was generalized in (Ahmad *et al*., 2010). Similar concept of topological mapping was also used in (Nordin and Ali, 2009) to provide navigation and localization for visually

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impaired people. In (Ahmad *et al*., 2006), a new method for mapping high dimensional signal, namely EEG into a low dimensional space (MC) has been developed. The whole process of this novel model consists of three main parts. The first part is flattening the EEG where the transformation of three dimensional space into two dimensional space that involved location of sensor in patients head with EEG signal (Fig. 1). This flattening process can preserves magnitude and orientation of the surface (Ahmad *et al*., 2006). Secondly, the EEG is processed using Fuzzy C-Means (FCM).

Finally, the optimal number of clusters is determined using cluster validity analysis. This new model enables tracking of brainstorm during seizure (Ahmad *et al*., 2006). Figure 2 are examples of Flat EEG. Red dots represent the electrodes while green dots

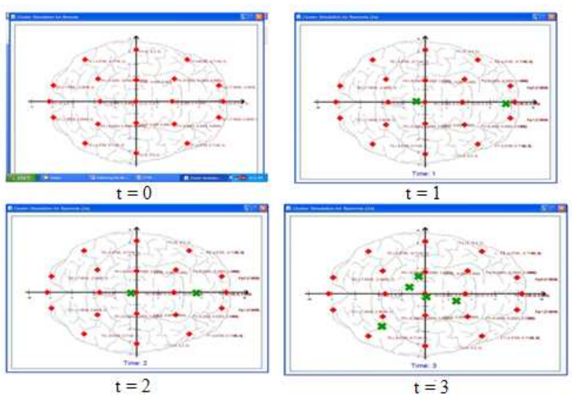

Fig. 2: Samples of Flat EEG

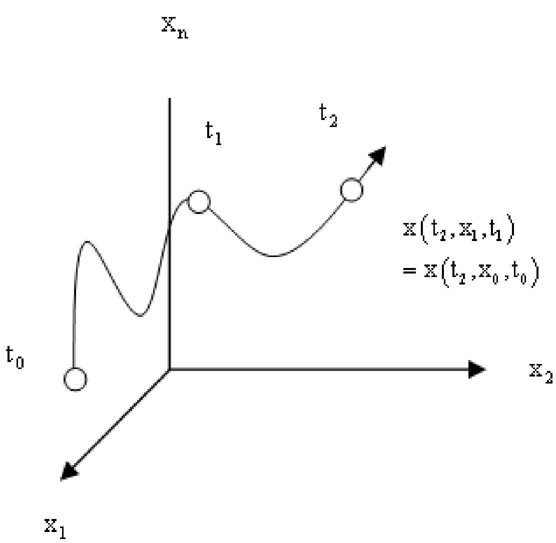

Fig 3: State space trajectory of seizure

represent the cluster centers after the transformation from the scalp of the patients.

On the other hand, seizure was modeled as a continuous dynamical system in (Ahmad *et al*., 2005) by assuming that it is governed by a set of n scalar differential equation with a solution of the form $w(t,\alpha,\beta)$. For a particular initial state and initial time, the motion, i.e., state space trajectory is written as $f: T \to R^n$ (Fig. 3) which is $w(t,\alpha_0,\beta_0)$. Besides, the augmented dynamic trajectory, $S=\{(w_1,\ldots,w_t t): w_i, t \in R\}$(denoted as X_t in (Ahmad *et al*., 2005), but we rename it as S to show that it is the augmented trajectory of seizure) that resulted when the motion is defined over an interval of time was also proven to exhibit linear ordering properties under the relation induced by the motion, f.

More research has been carried out on Flat EEG, in (Faisal and Tahir, 2010) for instance, Flat EEG on MC was presented as an algebraic structure. In (Faisal and Tahir, 2010), MC is rewrite as square matrices and transformed into upper triangular matrices using QR-Schur decomposition and finally as a semigroup of upper triangular matrices under matrix multiplication.

However, in this study the transformation of the dynamicity of seizure to visual platform, namely Flat EEG will be discussed and presented.

MATERIALS AND METHODS

We start as in (Ahmad *et al*., 2005) to model our series of Flat EEG as a dynamical system. Assuming that it is a continuous dynamical system and governed by a set of m+1 scalar differential equation with a solution of the form $e(t,\lambda,\gamma)$. For a given initial state and initial time we can write the motion as $g: T \to R^{m+1}$ which is $e(t,\lambda_0,\gamma_0)$. Hence, when we define over an interval of time, the motion produces a set of points known as the augmented dynamic trajectory which can be written as:

$$S_{\text{Flat EEG}} = \left\{ \begin{matrix} \left(e_p(x_1,y_1,z_1),\ldots,e_p(x_m,y_m,z_m),k,t\right) \\ : t \in [0,\infty),\ m,k \in N \end{matrix} \right\}$$

where, for each time, t, $(e_p(x_1,y_1,z_1),\ldots,e_p(x_m,y_m,z_m),k,t)$ represents one Flat EEG with $e_p(x_i,y_i,z_i)$, the electrical potential recorded from sensor (x_i,y_i,z_i) and k as the number of cluster centers.

Notice that the motion g induce a temporal ordering, $\prec_g$ on $S_{\text{Flat EEG}}$ which can be formalize as:

$$\begin{matrix}\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_1,t_1\right)\\ \prec_g\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_2,t_2\right)\end{matrix}\Leftrightarrow t_1\prec t_2$$

This temporal ordering makes $\left(S_{Flat\ EEG},\prec_g\right)$ a linearly ordered set.

Lemma 1: $\left(S_{Flat\ EEG},\prec_g\right)$ is a linearly ordered set.

Proof: By using the theorem from (Ahmad *et al*., 2005) which states that every temporal ordering on an augmented dynamic trajectory is a linear ordering, then $\left(S_{Flat\ EEG},\prec_g\right)$ is a linearly ordered set. ■

Order isomorphism: Our interest now is to show $(S,\prec_f)$ order isomorphic to $\left(S_{Flat\ EEG},\prec_g\right)$. We will use the definition of order isomorphism given in (Steve, 2008). We rewrite the definition as (with no changes in meaning),

Definition 1: Let $(P,\leq_p)$ and $(Q,\leq_q)$ be two linearly ordered sets and h: $P\rightarrow Q$ a function, then $(P,\leq_p)$ is order isomorphic to $(Q,\leq_q)$ if h is bijective and for all $x,y\in P$ and $h(x),h(y)\in Q$, $x\leq_p y$ if and only if $h(x)\leq_Q h(y)$.

We start by introducing lemma 2 and theorem 1 which will serve as our tools to prove theorem 2.

Lemma 2: Let $s_1,s_2\in(S,\prec_f)$ and $u_1,u_2\in\left(S_{Flat\ EEG},\prec_g\right)$, then $s_1\prec_f s_2\Leftrightarrow u_1\prec_g u_2$

Proof: Ahmad *et al*. (2005), the ordering relation induced by f gives the following mathematical statement

$$\left(w_1,...,w_n.t_1\right)\prec_f\left(v_1,...,v_n,t_2\right)\Leftrightarrow t_1\prec t_2$$

and as explained before, we know the motion g also induce a temporal ordering, $\prec_g$ on $S_{Flat\ EEG}$ which can be formalize as:

$$\begin{matrix}\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_1,t_1\right)\\ \prec_g\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_2,t_2\right)\end{matrix}\Leftrightarrow t_1\prec t_2$$

Now, substituting the former into the latter, we have:

$$\begin{aligned}&\left(w_1,...,w_n.t_1\right)\prec_f\left(v_1,...,v_n,t_2\right)\\ \Leftrightarrow&\begin{matrix}\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_1,t_1\right)\\ \prec_g\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_2,t_2\right)\end{matrix}\end{aligned}$$

Thus, the lemma is proven. ■

We now construct the required function and show that it is a bijection. The function mentioned is θ: $S\rightarrow S_{Flat\ EEG}$ and are defined as:

$$\begin{aligned}&\theta\left(w_1,...,w_n,t\right)\\ &=\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k,t\right)\end{aligned}$$

Theorem 1: The function $\theta:S\rightarrow S_{Flat\ EEG}$ defined as

$$\begin{aligned}&\theta\left(w_1,...,w_n,t\right)\\ &=\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k,t\right)\end{aligned}$$

is a bijection.

Proof:

Function: Suppose $w_i = v_i$ for $i = 1,2,3,\ldots n$ and $t_1 = t_2$

Then, $\left(w_1,...,w_n,t_1\right)=\left(v_1,...,v_n,t_2\right)$

Now, from lemma 2, this implies:

$$\begin{aligned}&\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_1,t_1\right)\\ &=\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_2,t_2\right)\end{aligned}$$

Thus, θ is a function.

Injective:

Suppose $\begin{aligned}&\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_1,t_1\right)\\ &=\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k_2,t_2\right)\end{aligned}$

Again by the same lemma:

$$\left(w_1,...,w_n,t_1\right)=\left(v_1,...,v_n,t_2\right)$$

Implying, $w_i=v_i$ for $i=1,2,3,...n$ and $t_1=t_2$

Thus, θ is injective.

Surjective: For all

$$\left(e_p\left(x_1,y_1,z_1\right),...,e_p\left(x_m,y_m,z_m\right),k,t\right)\in S_{Flat\ EEG}$$

There exists:

$$\left(w_1,...,w_n,t\right)\in S$$

Such that:

$$\theta(w_1,...,w_n,t) = \left(e_p(x_1,y_1,z_1),...,e_p(x_m,y_m,z_m),k,t\right)$$

Therefore, θ is surjective.

Since θ is both an injective and surjective, therefore the function θ is bijective.

Theorem 2: $(S,\prec_f)$ is order isomorphic to $(S_{Flat\ EEG},\prec_g)$

Proof: From lemma 2, we have:

$$(w_1,...,w_n.t_1)\prec_f (v_1,...,v_n,t_2) \Leftrightarrow \begin{array}{l} \left(e_p(x_1,y_1,z_1),...,e_p(x_m,y_m,z_m),k_1,t_1\right) \\ \prec_g \left(e_p(x_1,y_1,z_1),...,e_p(x_m,y_m,z_m),k_2,t_2\right) \end{array}$$

Substituting the function θ (from theorem 1) into above produces:

$$(w_1,...,w_n.t_1)\prec_f (v_1,...,v_n,t_2) \Leftrightarrow \theta(w_1,...,w_n.t_1)\prec_g \theta(v_1,...,v_n,t_2)$$

Thus, $(S,\prec_f)$ is order isomorphic to $(S_{Flat\ EEG},\prec_g)$ as desired. ■

Linearly ordered hausdorff topological space: In the following, we will show that the motions of seizure and Flat EEG that are modeled as a dynamical system is linearly ordered Hausdorff topological space. The ordering relation that will be used in constructing the interval topology is the induced strict total order (irreflexive, asymmetric and transitive) which we will denote it as $\prec_g^\bullet$. We start by introducing a result obtained from (Kopperman *et al*., 1998).

Corollary 1 (Kopperman *et al*., 1998): If τ is the topology of a well-formed space X, the statements

- X is a GO space
- X is Hausdorff

are equivalent.

Theorem 3: (S,τ_S) is a linearly ordered Hausdorff topological space.

Proof: Define a subbasis for set S as the collection of all order-open rays (under the induced strict order, $\prec_g^\bullet$):

$$\left\{\{w\in S \mid w\prec_f^\bullet u\},\{w\in S\mid u\prec_f^\bullet w\},\varphi,S\ \ \forall u\in S\right\}$$

this will then generate the following basis:

$$B=\left\{\begin{array}{l}\{w\in S\mid w\prec_f^\bullet u\},\{w\in S\mid u\prec_f^\bullet w\},\\ \{w\in S\mid u\prec_f^\bullet w\prec_f^\bullet v\},\phi,S\ \ \forall u,v\in S\end{array}\right\}$$

and eventually, we obtained the interval topology:

$$\tau_S=\left\{U_i \mid U_i=\bigcup_{i\in I}B_i \ni B_i\in B\right\}$$

This generated interval topology makes the pair (S,τ_S) a linearly ordered topological space. Note that, if the topology of an ordered space X is generated by collection of rays, then it is called a well-formed space (Kopperman *et al*., 1998). Using this fact, (S,τ_S) is then a well-formed space. Together with the fact from (Bennett and Lutzer, 1996) which says that the class of GO-spaces coincides with the class of subspaces of LOTS, we can therefore say (S,τ_S) is a GO-space, by viewing (S,τ_S) as a subspace of the LOTS (S,τ_S). By using corollary 1, we can conclude that S is Hausdorff. Thus, (S,τ_S) is a linearly ordered Hausdorff topological space. ■

As the motion of seizure is a Hausdorff LOTS, therefore any event during seizure, introduced in (Ahmad *et al*., 2004), can be characterized by an open set from the LOTS. In other words, no matter how close two events are, it still can be differentiated by two unique disjoint open sets. Therefore, we provide the following lemma.

Lemma 3: Any event of seizure can be characterized uniquely by an open set from its Lots.

Theorem 4: $(S_{Flat\ EEG},\tau_{S_{Flat\ EEG}})$ is a linearly ordered Hausdorff topological space.

Proof: Use similar proof as in theorem 3. ■

Homeomorphism: Now, we will show that the function θ is a homeomorphism and (S,τ_S) is homeomorphic to $(S_{Flat\ EEG},\tau_{S_{Flat\ EEG}})$.

Since an interval topology is generated by subbasis, any open set can be written as a union of finite intersections of elements of the subbasis. Therefore, to prove the continuity of a function, it is suffices to show that the inverse image of each subbasis element is open (James, 2000).

Theorem 5: The function θ:S$\rightarrow S_{\text{Flat EEG}}$ defined as:

$$\theta(w_1,...,w_n,t) \\ =(e_p(x_1,y_1,z_1),...,e_p(x_m,y_m,z_m),k,t)$$

is a continuous function.

Proof:

Case 1: Open sets in the form $\{q \in S_{\text{Flat EEG}} \mid q \prec_g^{\bullet} c\}$, i.e., an order-open ray clearly:

$$\theta^{-1}\left(\{q \in S_{\text{Flat EEG}} \mid q \prec_g^{\bullet} c\}\right) = \{p \in S \mid p \prec_f^{\bullet} a\}$$

for some $a \in S$ since θ^{-1} is bijective

From the constructed interval topology for S, any order-open ray in S is an open set in S

Now, since $\{p \in S \mid p \prec_f^{\bullet} a\}$ is an order-open ray

Thus, $\{p \in S \mid p \prec_f^{\bullet} a\}$ is an open set in S

Case 2: Open sets in the form $\{q \in S_{\text{Flat EEG}} \mid c \prec_g^{\bullet} q\}$, i.e., an order-open ray clearly:

$$\theta^{-1}\left(\{q \in S_{\text{Flat EEG}} \mid c \prec_g^{\bullet} q\}\right) = \{p \in S \mid a \prec_f^{\bullet} p\}$$

for some $a \in S$ since θ^{-1} is bijective

From the constructed interval topology for S, any order-open ray in S is an open set in S

Now, since $\{p \in S \mid a \prec_f^{\bullet} p\}$ is a order-open ray

Thus, $\{p \in S \mid a \prec_f^{\bullet} p\}$ is an open set in S

Case 3: The whole set $S_{\text{Flat EEG}}$ and empty set φ.

$\theta^{-1}\{S_{\text{Flat EEG}}\} = S$ and $\theta^{-1}\{\theta\} = \theta$ since θ^{-1} is bijective

From the constructed interval topology for S, the set S itself and the empty set θ is open.

Combining all these three cases, the inverse image of each subbasis element is open. Thus,

$$\theta(w_1,...,w_n,t) \\ =(e_p(x_1,y_1,z_1),...,e_p(x_m,y_m,z_m),k,t)$$

is a continuous function that maps from (S,τ_S) to $\left(S_{\text{Flat EEG}},\tau_{S_{\text{Flat EEG}}}\right)$. ■

Theorem 6: The function $\theta^{-1}: S_{\text{Flat EEG}} \rightarrow S$ defined as:

$$\theta^{-1}\left(e_p(x_1,y_1,z_1),...,e_p(x_m,y_m,z_m),k,t\right) \\ =(w_1,...,w_n,t)$$

is continuous.

Proof: Use similar proof as for theorem 5. ■

Theorem 7: The function $\theta: S \rightarrow S_{\text{Flat EEG}}$ defined as:

$$\theta(w_1,...,w_n,t) \\ =(e_p(x_1,y_1,z_1),...,e_p(x_m,y_m,z_m),k,t)$$

is a homeomorphism.

Proof: Since θ is bijective (theorem 1), continuous (theorem 5) and its inverse is continuous (theorem 6) therefore, θ is a homeomorphism. ■

Corollary 2: (S,τ_S) is homeomorphic to $\left(S_{\text{Flat EEG}},\tau_{S_{\text{Flat EEG}}}\right)$.

Topologically conjugating: One usual way to relate two dynamical systems is with the topological notion of conjugacy (Erik and Joseph, 2010). Two dynamic systems are topologically conjugated if there exist a homeomorphsim h such that $h \circ d_1 = d_2 \circ h$, where d_1 and d_2 are the dynamic functions that act on the system (Erik and Joseph, 2010). Using this concept, we show that the dynamic functions that act on S and $S_{\text{Flat EEG}}$ share the same dynamics.

Firstly, we rewrite the dynamic function that act on S and $S_{\text{Flat EEG}}$ as f: $R^{n+1} \rightarrow R^{n+1}$ and g: $R^{m+1+1} \rightarrow R^{m+1+1}$ respectively by including the information of system state at that particular time into the domain and also the time, t into the range. This changes will not create any inconsistency as both functions operate only on the time, t. Thus, function f and g can respectively be defined as:

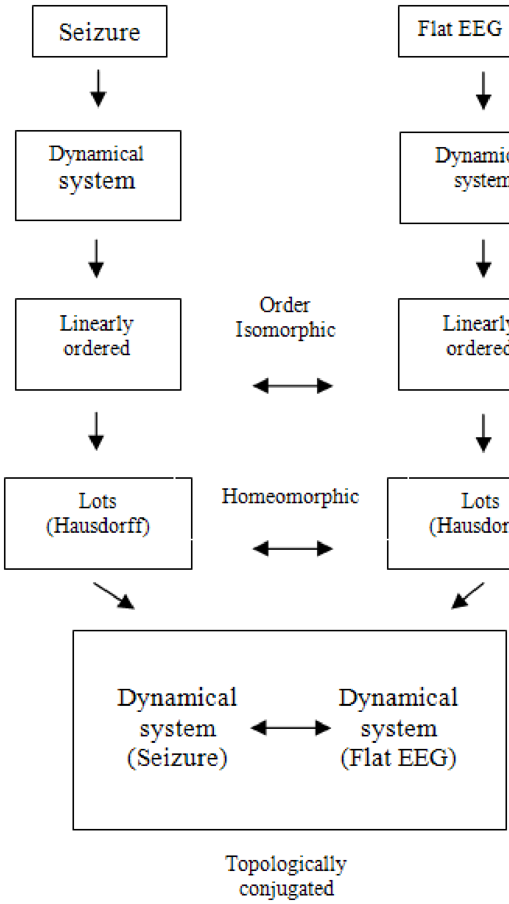

Fig. 4: Framework

$$f\left(w_1,\ldots,w_n,t_j\right)=\left(v_1,\ldots,v_n,t_{j+1}\right)$$

and:

$$g\left(e_p\left(x_1,y_1,z_1\right),e\ldots,e_p\left(x_m,y_m,z_m\right),k_j,t_j\right)$$
$$=\left(e_p\left(x_1,y_1,z_1\right),\ldots,e_p\left(x_m,y_m,z_m\right),k_{j+1},t_{j+1}\right)$$

Theorem 8: Seizure and Flat EEG that are modeled as a dynamical systems share the same dynamics (topologically conjugate).

Proof: Composition of functions θ and f is:

$$\theta[f(w_1,\ldots,w_n,t_j)]$$
$$=\theta\left(v_1,\ldots,v_n,t_{j+1}\right)$$
$$=\left(e_p\left(x_1,y_1,z_1\right),\ldots,e_p\left(x_m,y_m,z_m\right),k_{j+1},t_{j+1}\right)$$

Composition of functions g and θ is:

$$g[\theta(w_1,\ldots,w_n,t_j)]$$
$$=g\left(e_p\left(x_1,y_1,z_1\right),\ldots,e_p\left(x_m,y_m,z_m\right),k_j,t_j\right)$$
$$=\left(e_p\left(x_1,y_1,z_1\right),\ldots,e_p\left(x_m,y_m,z_m\right),k_{j+1},t_{j+1}\right)$$

This shows that $\theta\circ f=g\circ\theta$

Thus, the two dynamic systems with their respective function acting on them share the same dynamics. ■

Figure 4 portray the framework.

RESULTS

In this study, we have shown that seizure and Flat EEG that are modeled as a dynamical systems share the same dynamics. Besides, their augmented dynamic trajectory is linearly ordered and order isomorphic to each other by the relation induced from their motion. By endowing the interval topology, the LOTS are proven to be Hausdorff and homeomorphic to each other. Additionally, we show that any event of seizure can be characterized uniquely by an open set from its LOTS.

DISCUSSION

We have shown that seizure and Flat EEG share the same dynamics by using the concept of topological conjugacy. Therefore, the dynamics and orders of seizure are embedded in EEG signals.

CONCLUSION

In this study, we linked seizure and Flat EEG from few aspects. By modeling Flat EEG as a continuous dynamical system, we composed it into a set of points, $S_{\text{Flat EEG}}$ that exhibit linear ordering properties. A function, θ is then introduced to show that the set of points are order isomorphic to S and homeomorphic when endowed with the interval topology. Finally, we show that seizure and Flat EEG that are modeled as a dynamical systems share the same dynamics by using the same function.

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REFERENCES

Ahmad, T., J. Abdullah, F. Fauziah, F. Mustapha and H.A.M.H. Zabidi, 2006. Tracking the storm in the brain. J. Quantitative Methods, 2: 1-9.

Ahmad, T., L.Y. Liau and S.A. Rashdi, 2004. Algebra of Time of Epilepsy Disorder Patient during Seizure. Prosiding Simposium Kebangsaan Sains Matematik Ke-12 (SKSM-12), 23rd-24th December. UIAM, pp: 1-7.

Ahmad, T., R.S. Ahmad, W.E.Z.W.A. Rahman, L.L., Yun and F. Zakaria, 2005. Homeomorphism of Fuzzy Topographic Topological Mapping (FTTM). Matematika, 21: 35-42.

Ahmad, T., R.S. Ahmad, W.E.Z.W.A. Rahman, L.L., Yun and F. Zakaria, 2008. Fuzzy Topographic Topological Mapping For Localization Simulated Multiple Current Sources of MEG. J. Interdis. Math., 11: 381-393.

Ahmad, T., S.S. Jamian and J. Talib, 2010. Generalized Finite Sequence of Fuzzy Topographic Topological Mapping. J. Math. Stat., 6: 151-156. DOI: 10.3844/.2010.151.156

Ahmad, T., Z. Fauziah, J. Abdullah and F. Mustapha, 2005. Dynamical system of an epileptic seizure. Proceeding of the Asian Conference on Sensors and International Conference on New Techniques in Pharmaceutical and Biomedical Research, Kuala Lumpur, 6th-7th September. IEEE Catalog Number: 05EX11610. ISBN: 0-7803-9371-6, pp: 1-4.

Bennett, H.R. and D.J. Lutzer, 1996. Point-countability on generalized ordered spaces. Topol. Appl., 20: 1-17. http://www.math.wm.edu/~lutzer/papers/PointCtblBases.pdf

Erik, M.B. and D.S. Joseph, 2010. On Comparing Dynamical Systems by Defective Conjugacy: A Symbolic Dynamics Interpretation of Commuter Functions. Elsevier, Physica D., 239: 579-590. http://sciencia.org/stories/428269/On_comparing_dynamical_systems_by_defective_conjugacy_A_symbolic_dynamics_interpretation_of_commuter_functions.html

Faisal, A.M.B. and A. Tahir, 2010. EEG signals during epileptic seizure as a semigroup of upper triangular matrices. Am. J. Applied Sci., 7: 540-544. ISSN: 1546-9239

James, R.M., 2000. Topology, 2nd Edn., Pearson, Prentice Hall, Chapter 2, pp: 102.

Kopperman, R.D., E.H. Kronheimer and R.G. Wilson, 1998. Topologies on totally ordered sets. Elsevier, Topol. Appli., 90: 165-185.

Nordin, M.J. and A.M. Ali, 2009. Indoor Navigation and Localization for Visually impaired people Using Weighted Topological Map. J. Comput. Sci., 5: 883-889. DOI: 10.3844/.2009.883.889

Steve, R., 2008. Lattices and Ordered Sets, 1st Edn., Springer, Chapter 1, pp: 13.