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A Certain Subclass of Meromorphically Multivalent Analytic Functions with Negative Coefficients

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Abstract: The paper aims to introduce a certain subclass $S^*_g(A,B,\alpha,p,j)$ of the class $\Sigma^*(p)$ of meromorphically multivalent functions with negative coefficients, defined by using the definitions of Hadamard product and subordination for two functions belong to the class $\Sigma^*(p)$. We first investigate the geometric characterization property, giving the coefficient estimates for functions in the class $S^*_g(A,B,\alpha,p,j)$. We also obtain the distortion theorem, radii of meromorphically p-valent starlikeness and convexity of order ($0\le\gamma<p$), neighborhood property, partial sums, convolution properties as well as integral operator and integral representation.

Keywords: Meromorphic Function, Multivalent Function, Subordination, Hadamard Product

Introduction

Let $\Sigma(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \left(p \in \mathbb{N} := \{1,2,3\ldots\}\right) \quad (1.1)$$

Which are analytic and p-valent (multivalent) in the punctured unit disk $U^*: = U^*(1)$ where.

$$U^*(r) = \{z : 0 < |z| < r(0 < r \le 1)\} = U(r)\setminus\{0\}$$

Let $\Sigma^*(p)$ denote the subclass of $\Sigma(p)$ consisting of functions of the form:

$$f(z) = z^{-p} - \sum_{n=p}^{\infty} a_n z^n \left(a_n \ge 0; p \in \mathbb{N}\right) \quad (1.1)$$

Which are analytic and p-valent (multivalent) in the punctured unit disk U^*.

A function $f\in\Sigma^*(p)$ is said to be meromorphically p-valent starlike of order ($0\le\gamma<p$) in $U^*(r)$ if $\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \gamma, \left(z \in U^*(r); 0 < r \le 1; 0 \le \gamma < p\right)$, also a function $f\in\Sigma^*(p)$ is said to be meromorphically p-valent convex of order ($0\le\gamma<p$) in $U^*(r)$ if $\text{Re}\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \gamma\left(z \in U^*(r); 0 < r \le 1; 0 \le \gamma < p\right)$ (cf. e.g., Duren, 1983; Goodman, 1983; Srivastava and Owa, 1992). Let B be a subclass of the class $\Sigma^*(p)$. We define the radius of meromorphically p-valent starlike of order γ and the radius of meromorphically p-valent convex of order γ for the class B by:

$$\mathcal{R}^*_\gamma = \inf_{f\in B}(\sup\sup\{r \in (0,1] : f \text{ is meromorphically } p - \text{valent starlike of order } \gamma \text{ in } U^*(r)\}$$

$$\mathcal{R}^c_\gamma = \inf_{f\in B}(\sup\sup\{r \in (0,1] : f \text{ is meromorphically } p - \text{valent convex of order } \gamma \text{ in } U^*(r)\},$$

respectively.

The convolution or the Hadamard product of two meromorphic p-valent functions f and g, where f is given by (1.1) and $g(z) = z^{-p} - \sum_{n=p}^{\infty} b_n z^n \left(b_n \ge 0; p \in \mathbb{N}\right)$ is denoted by $f*g$ and defined by (Aouf, 2009):

$$(f * g)(z) = z^{-p} - \sum_{n=p}^{\infty} a_n b_n z^n \quad (1.2)$$

Recall (Aouf and El-Ashwah, 2009) that an analytic function f is said to be subordinate to an analytic function g written $f \prec g$, if $f(z) = g(w(z)), |z| < 1$ for some analytic function w with $|w(z)|<1$.

The significance of the geometric properties and characteristics of various interesting subclasses of the class $\Sigma(p)$ of meromorphically univalent and multivalent functions with negative coefficients or positive coefficients has comprehensively been investigated by many well-known complex analysts. For instance, (Uralegaddi and Ganigi 1987; Srivastava *et al*., 1998; Aouf and Hossen, 1993; Liu and Srivastava, 2004; Aouf and Shammaky, 2005; Aouf, 2009; Aouf and El-Ashwah, 2009; Aouf, 2010; Kamali *et al*., 2011; Makinde, 2012).

By making use of the following definition for Hadamard product and subordination, a new subclass $\mathcal{S}_g^*(A,B,\alpha,p,j)$ of functions in $\Sigma^*(p)$ is introduced.

Definition 1.1. A function $f(z) \in \Sigma^*(p)$ is said to be in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$ if it satisfies the following subordination condition:

$$\frac{z\left((f*g)(z)\right)^{(j+1)}}{((f*g)(z))^{(j)}} \prec -\left[\frac{(p+j)+\left[B(p+j)-(p+j-\alpha)(A-B)\right]z}{1+Bz}\right] \tag{1.3}$$

where, $-1 \le A < B \le 1$; $0 < B \le 1$; $0 \le \alpha < p+j$; $j \le p \le n$,; $p \in \mathbb{N}$; $j \in 2\mathbb{N} \cup \{0\} = \{0, 2, 4, \ldots\}$ $z \in U^*$. or, equivalently, if:

$$\left|\frac{\dfrac{z((f*g)(z))^{(j+1)}}{((f*g)(z))^{(j)}} + (p+j)}{B\dfrac{z((f*g)(z))^{(j+1)}}{((f*g)(z))^{(j)}} + \left[B(p+j)-(p+j-\alpha)(A-B)\right]}\right| < 1 \tag{1.4}$$

The aim intended to be achieved in the current analysis is to identify coefficient estimates, distortion theorem, radii of meromorphically p-valent star likeness and convexity of order $(0 \le \gamma < p)$, neighborhood property, partial sums. Moreover, the convolution properties and integral operator and integral representation are investigated.

Upper Bounds

In the following section, we establisha characterization property which provides a necessary, sufficient condition for a function $f(z)$, defined by (1.1), belongs to the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$ and obtain the coefficient estimates.

Theorem 1. Let the function $f(z)$ be given by (1.1). Then $f(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)$ if and only if:

$$\sum_{n=p}^{\infty} \delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right] a_n b_n \le (p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!} \tag{2.1}$$

Where:

$$\delta(n,j) = \frac{n!}{(n-j)!} = \begin{cases} n(n-1)\ldots(n-j+1) & (j \ne 0) \\ 1 & (j=0) \end{cases} \tag{2.2}$$

Proof. Suppose that the function $f(z)$ defined by (1.1) is in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$. Then from (1.1) and (1.4), we have:

$$\left|\frac{z(f*g)^{(j+1)}(z)+(p+j)(f*g)^{(j)}(z)}{Bz(f*g)^{(j+1)}(z)+[B(p+j)-(p+j-\alpha)(A-B)](f*g)^{(j)}(z)}\right|$$

$$= \left|\frac{-\sum_{n=p}^{\infty}\delta(n,j)(n+p)a_n b_n z^{n+p}}{(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!} - \sum_{n=p}^{\infty}\delta(n,j)\left[B(n+p)-(p+j-\alpha)(B-A)\right]a_n b_n z^{n+p}}\right| < 1 \left(z \in U^*\right) \tag{2.2}$$

The choice of z to be real and letting $z \to 1^-$ through real value when $|Re\{z\}| \le |z|$ for any z, then the inequality (2.2) directly gives the desired condition in (2.1). Conversely, assume that the condition (2.1) holds true and let $|z| = 1$, then we have:

$$\left|\frac{z(f*g)^{(j+1)}(z)+(p+j)(f*g)^{(j)}(z)}{Bz(f*g)^{(j+1)}(z)+[B(p+j)-(p+j-\alpha)(A-B)](f*g)^{(j)}(z)}\right|$$

$$\le \frac{\sum_{n=p}^{\infty}\delta(n,j)(n+p)a_n b_n}{(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!} - \sum_{n=p}^{\infty}\delta(n,j)\left[B(n+p)-(p+j-\alpha)(B-A)\right]a_n b_n} < 1 \left(z \in U^*\right) \tag{2.3}$$

By hypothesis. Hence, by the maximum modulus theorem $f(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)$.

Corollary 1. Let *f*(*z*) be defined by (1.1). If $f(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)$, then:

$$a_n \le \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]b_n} \quad (2.4)$$

$$\left(j \le p \le n, j \in 2\mathbb{N}\cup\{0\}, p \in \mathbb{N}\right)$$

The result is sharp for the function *f*(*z*) given by:

$$f(z) = z^{-p} - \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\begin{bmatrix}(1+B)(n+p)\\+(p+j-\alpha)(B-A)\end{bmatrix}b_n} z^n \quad (2.5)$$

$$\left(j \le p \le n, j \in 2\mathbb{N}\cup\{0\}, p \in \mathbb{N}\right)$$

Distortion Theorem

The following theorem proves the distortion inequality for the function $f \in \mathcal{S}_g^*(A,B,\alpha,p,j)$. Such property for another class is investigated by many researchers among them Aouf and El-Ashwah (2009).

Theorem 2: If a function *f*(*z*) given by (1.1) is in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$. Then:

$$\frac{(p+q-1)!}{(p-1)!}r^{-p-q} - \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\frac{(p-q)!}{(p-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p} r^{n-q} \le \left|f^{(q)}(z)\right|$$

$$\le \frac{(p+q-1)!}{(p-1)!}r^{-p-q} + \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\frac{(p-q)!}{(p-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p} r^{n-q} \quad (3.1)$$

$$\left(0<|z|=r<1; 0 \le q \le j \le p; q \in \mathbb{N}_0; j \in 2\mathbb{N}\cup\{0\}; p \in \mathbb{N}\right)$$

Provided $b_n \ge b_p \ge 1 (n \ge p)$. The result is sharp for the functions *f* given by:

$$f(z) = z^{-p} - \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p} z^p \quad (3.2)$$

$$\left(j \le p \le n, j \in 2\mathbb{N}\cup\{0\}, p \in \mathbb{N}\right)$$

Proof. Since $f(z) = z^{-p} - \sum_{n=p}^{\infty} a_n z^n$ is in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$ and from Theorem 1 together with:

$$\frac{(p-q)!}{(p-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p \le \frac{(n-q)!}{(n-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_n \,(n \ge p)$$

We have:

$$\frac{(p-q)!}{(p-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p \sum_{n=p}^{\infty} \frac{n!}{(n-q)!}a_n \le (p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}$$

Or:

$$\sum_{n=p}^{\infty} \frac{n!}{(n-q)!}a_n \le \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\frac{(p-q)!}{(p-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p}$$

Now by differentiating $f(z) = z^{-p} - \sum_{n=p}^{\infty} a_n z^n (q)$ times, we have:

$$f^{(q)}(z) = (-1)^q \frac{(p+q-1)!}{(p-1)!}z^{-p-q} - \sum_{n=p}^{\infty} \frac{n!}{(n-q)!} a_n z^{n-q} \left(j \le p \le n, j \in 2\mathbb{N}\cup\{0\}, p \in \mathbb{N}\right)$$

We have for $|z| = r<1$.

$$|f^{(q)}(z)| \le \frac{(p+q-1)!}{(p-1)!}r^{-p-q} + r^{n-q}\sum_{n=p}^{\infty} \frac{n!}{(n-q)!}a_n r^{n-q}$$

$$\le \frac{(p+q-1)!}{(p-1)!}r^{-p-q} + r^{n-q} \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\frac{(p-q)!}{(p-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p}$$

And similarly:

$$|f^{(q)}(z)| \geq \frac{(p+q-1)!}{(p-1)!} r^{-p-q} - r^{n-q}$$

$$\frac{(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}{\dfrac{(p-q)!}{(p-j)!}\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p}$$

The sharpness of the inequality in (3.1) satisfies by the function *f*(*z*) given by (3.2).

Radii of Starlikeness and Convexity

This section considers the radii of meromorphically p-valent starlikeness of order $\gamma(0\le\gamma<p)$ and meromorphically p-valent convexity of order $\gamma(0\le\gamma<p)$ for the functions that belong to the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$, by using methods applied by Kamali *et al*. (2011) and others.

Theorem 3: Let $f(z)\in\mathcal{S}_g^*(A,B,\alpha,p,j)$. Then:

f is meromorphically *p*-valent starlike of order $\gamma(0\le\gamma<p)$ in $|z|<r_2$, where:

$$r_1 = \inf_{n\ge p}\left\{\frac{(p-\gamma)\delta(n,j)\begin{bmatrix}(1+B)(p+n)\\+(p+j-\alpha)(B-A)\end{bmatrix}b_n}{(n+\gamma)(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}\right\}^{\frac{1}{n+p}} \quad (4.1)$$

f is meromorphically *p*-valent convex of order $\gamma(0\le\gamma<p)$ in $|z|<r_2$, where:

$$r_2 = \inf_{n\ge p}\left\{\frac{p(p-\gamma)\delta(n,j)\begin{bmatrix}(1+B)(p+n)\\+(p+j-\alpha)(B-A)\end{bmatrix}b_n}{n(n+\gamma)(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}\right\}^{\frac{1}{n+p}} \quad (4.2)$$

Each of these results is sharp for the function *f*(*z*) given by (2.5).

Proof. (i) From the definition (1.1), we obtain:

$$\left|\frac{\dfrac{zf'(z)}{f(z)}+p}{\dfrac{zf'(z)}{f(z)}-p+2\gamma}\right| \le \frac{\sum_{n=p}^{\infty}(n+p)a_n|z|^{n+p}}{2(p-\gamma)-\sum_{n=p}^{\infty}(n-p+2\gamma)a_n|z|^{n+p}}$$

Thus, we have the desired inequality:

$$\left|\frac{\dfrac{zf'(z)}{f(z)}+p}{\dfrac{zf'(z)}{f(z)}-p+2\gamma}\right| \le 1 (0\le\gamma<p; p\in\mathbb{N})$$

If:

$$\sum_{n=p}^{\infty}\frac{(n+\gamma)}{(p-\gamma)}a_n|z|^{n+p}\le 1 \quad (4.3)$$

So, by Theorem 1, the condition (4.3) will be true if:

$$\frac{(n+\gamma)}{(p-\gamma)}|z|^{n+p} \le \frac{\delta(n,j)\begin{bmatrix}(1+B)(p+n)\\+(p+j-\alpha)(B-A)\end{bmatrix}b_n}{(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}, (p\le n; p\in\mathbb{N})$$

Therefore:

$$|z| \le \left\{\frac{(p-\gamma)\delta(n,j)\begin{bmatrix}(1+B)(p+n)\\+(p+j-\alpha)(B-A)\end{bmatrix}b_n}{(n+\gamma)(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}\right\}^{\frac{1}{n+p}} \quad (4.4)$$

Putting $|z| = r_1$ in (4.4), we get the radius of starlikeness.

In order to prove the second assertion of Theorem 3, we find from the definition (1.1) that:

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}+p}{1+\dfrac{zf''(z)}{f'(z)}-p+2\gamma}\right| \le \frac{\sum_{n=p}^{\infty}n(n+p)a_n|z|^{n+p}}{2p(p-\gamma)-\sum_{n=p}^{\infty}n(n-p+2\gamma)a_n|z|^{n+p}}$$

Thus, we have the desired inequality:

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}+p}{1+\dfrac{zf''(z)}{f'(z)}-p+2\gamma}\right| \le 1, (0\le\gamma<p; p\in\mathbb{N})$$

If:

$$\sum_{n=p}^{\infty}\frac{n(n+\gamma)}{p(p-\gamma)}a_n|z|^{n+p}\le 1 \quad (4.5)$$

Hence, by Theorem 1, the condition (4.5) will be satisfied if:

$$\frac{n(n+\gamma)}{p(p-\gamma)}|z|^{n+p} \le \frac{\delta(n,j)\left[(p+j-\alpha)(B-A)+(1+B)(p+n)\right]b_n}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}.$$

$(p \le n, p \in \mathbb{N})$

Therefore:

$$|z| \le \left\{ \frac{p(p-\gamma)\delta(n,j)\begin{bmatrix}(p+j-\alpha)\\(B-A)+(1+B)(p+n)\end{bmatrix}b_n}{n(n+\gamma)(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} \right\}^{\frac{1}{n+p}} \quad (4.6)$$

Setting $|z| = r_1$ in (4.6), we obtain the radius of convexity, which completes the proof of Theorem 3.

Neighborhood Property

Depending on the earlier works by (Goodman, 1957; Ruscheweyh, 1981; Liu and Srivastava, 2004; Aouf, 2009; Aouf and El-Ashwah, 2009) that based upon the familiar concept of neighborhood of analytic functions, we introduce the definition of the δ-neighborhood of a function $f(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)$ by:

$$N_\delta(f) = \left\{ \begin{array}{l} \phi \in \mathcal{S}_g^*(A,B,\alpha,p,j) : \phi(z) = z^{-p} - \sum_{n=p}^{\infty} d_n z^n \\ and \sum_{n=p}^{\infty} \frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]b_n}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} \\ |a_n - d_n| \le \delta \end{array} \right\}$$

$$\left(\begin{array}{l} -1 \le A < B \le 1; 0 < B \le 1; 0 \le \alpha < p \\ -j; j \le p \le n; p \in N,; j \in 2\mathbb{N}\mathbb{U}\{0\}; \delta > 0 \end{array} \right)$$

Theorem 4: If $f(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)$ satisfies the following condition $\frac{f(z)+\sigma z^{-p}}{1+\sigma} \in \mathcal{S}_g^*(A,B,\alpha,p,j)$ for any complex number σ,|σ| = 1 such that |σ|<δ, then $N_\delta(f) \subset \mathcal{S}_g^*(A,B,\alpha,p,j)$.

Proof. It is easily obvious that $f(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)$ if and only if:

$$\left| \frac{z(f*g)^{(j+1)}(z)+(p+j)(f*g)^{(j)}(z)}{Bz(f*g)^{(j+1)}(z)+[B(p+j)-(p+j-\alpha)(A-B)](f*g)^{(j)}(z)} \right| < 1$$

For any complex number ε with |ε| =, we have:

$$\frac{z(f*g)^{(j+1)}(z)+(p+j)(f*g)^{(j)}(z)}{Bz(f*g)^{(j+1)}(z)+[B(p+j)+(p+j-\alpha)(B-A)](f*g)^{(j)}(z)} \ne \varepsilon$$

Consequently:

$$1 - \sum_{n=p}^{\infty} \frac{\delta(n,j)\begin{bmatrix}(\varepsilon B-1)(n+p)\\+\varepsilon(p+j-\alpha)(B-A)\end{bmatrix}a_n b_n}{\varepsilon(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} z^{n+p} \ne 0$$

Which is equivalent to:

$$z^{-p} - \sum_{n=p}^{\infty} \frac{\delta(n,j)\left[(\varepsilon B-1)(n+p)+\varepsilon(p+j-\alpha)(B-A)\right]a_n b_n}{\varepsilon(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} z^n \ne 0$$

Thus, $f(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)$ if and only if $\frac{(f*h)(z)}{z^{-p}} \ne 0 (z \in U^*)$, where $h(z) = z^{-p} - \sum_{n=p}^{\infty} c_n z^n$ and $c_n = \frac{\delta(n,j)\left[(\varepsilon B-1)(n+p)+\varepsilon(p+j-\alpha)(B-A)\right]b_n}{\varepsilon(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}$, we have $|c_n| \le \frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]b_n}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}$.

Since $\frac{f(z)+\sigma z^{-p}}{1+\sigma} \in \mathcal{S}_g^*(A,B,\alpha,p,j)$, therefore $\frac{\left(\frac{f(z)+\sigma z^{-p}}{1+\sigma} * h(z)\right)}{z^{-p}} \ne 0$, which is equivalent to:

$$\frac{(f*h)(z)}{(1+\sigma)z^{-p}} + \frac{\sigma}{1+\sigma} \ne 0 \quad (5.1)$$

Now suppose that $\left|\frac{(f*h)(z)}{z^{-p}}\right| < \delta$ Then by (5.1), we have:

$$\left| \frac{(f*h)(z)}{(1+\sigma)z^{-p}} + \frac{\sigma}{1+\sigma} \right| \ge \frac{|\sigma|}{|1+\sigma|} - \frac{1}{|1+\sigma|}\left|\frac{(f*h)(z)}{z^{-p}}\right| > \frac{|\sigma|-\delta}{|1+\sigma|} \ge 0$$

This is contradiction by |σ|<δ and however, we have $\left|\frac{(f*h)(z)}{z^{-p}}\right| \ge \delta$. If $\phi(z) = z^{-p} - \sum_{n=p}^{\infty} d_n z^n \in N_\delta(f)$ then:

$$\delta - \left|\frac{(\phi*h)(z)}{z^{-p}}\right| \le \left|\frac{((\phi-f)*h)(z)}{z^{-p}}\right| \le \sum_{n=p}^{\infty} |a_n - d_n||c_n||z^n|$$

$$< \sum_{n=p}^{\infty} \frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]b_n}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} |a_n - d_n| \le \delta$$

Partial Sums

By following (Silvia, 1985; Silverman, 1997) previous investigations, we are able in this section to identify the bounds $\frac{f(z)}{s_{p+k}(z)}$ and $\frac{s_{p+k}(z)}{f(z)}$.

Theorem 5: Let $f(z)\in\Sigma^*(p)$ be in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$ and define the partial sums as follows:

$$s_1(z)=z^{-p}$$
$$s_m(z)=z^{-p}-\sum_{n=p}^{m}a_nz^n(m\geq p)$$

Also suppose that:

$$\sum_{n=p}^{\infty}d_na_n\leq 1\left(d_n=\frac{\delta(n,j)\begin{bmatrix}(1+B)(n+p)\\+(p+j-\alpha)(B-A)\end{bmatrix}b_n}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}\right)\quad(6.1)$$

Then we have:

$$\operatorname{Re}\left\{\frac{f(z)}{s_m(z)}\right\}>1-\frac{1}{d_{m+1}}\quad(6.2)$$

$$\operatorname{Re}\left\{\frac{s_m(z)}{f(z)}\right\}>\frac{d_{m+1}}{1+d_{m+1}}(z\in U^*)\quad(6.3)$$

The results in (6.2) and (6.3) are sharp for n.

Proof. We can see from (6.1) that $1<d_n<d_n+1$ $(n\in N)$. Since $\{d_n\}$ is an increasing sequence, we obtain:

$$\sum_{n=p}^{m}a_n+d_{m+1}\sum_{n=m+1}^{\infty}a_n\leq 1\quad(6.4)$$

By setting $g_1(z)=d_{m+1}\left(\frac{f(z)}{s_m(z)}-\left(1-\frac{1}{d_{m+1}}\right)\right)=1-\frac{d_{m+1}\sum_{n=p}^{\infty}a_nz^{n+p}}{1-\sum_{n=p}^{m}a_nz^{n+p}}$ from (6.4) we find that $\left|\frac{g_1(z)-1}{g_1(z)+1}\right|\leq\frac{d_{m+1}\sum_{n=p}^{\infty}a_nz^{n+p}}{2-2\sum_{n=p}^{m}a_n-d_{p+k}\sum_{n=p+k}^{\infty}a_n}\leq 1$.

This proves (6.2). If we take $f(z)=z^{-p}-\frac{z^{p+k}}{d_{m+1}}$ then $\frac{f(z)}{s_m(z)}=1-\frac{z^{2p+k}}{d_{m+1}}\to 1-\frac{1}{d_{m+1}}\,as\ z\to 1$ which shows that the bound in (6.2) is best possible.

Similary, if we let $g_2(z)=(1+d_{m+1})\left(\frac{s_m(z)}{f(z)}-\frac{d_{m+1}}{1+d_{m+1}}\right)=1+\frac{(1+d_{m+1})\sum_{n=m+1}^{\infty}a_nz^{n+p}}{1-\sum_{n=p}^{\infty}a_nz^{n+p}}$ and make use of (6.4), we obtain $\left|\frac{g_2(z)-1}{g_2(z)+1}\right|\leq\frac{(1+d_{m+1})\sum_{n=m+1}^{\infty}a_n}{2-2\sum_{n=p}^{m}a_n-(d_{m+1}-1)\sum_{n=m+1}^{\infty}a_n}\leq 1,(z\in U^*)$ which yields inequality (6.3). If we take $f(z)=z^{-p}-\frac{z^{p+k}}{d_{m+1}}$ then $\frac{s_m(z)}{f(z)}=\frac{d_{m+1}}{d_{m+1}-z^{2p+k}}\to\frac{d_{m+1}}{d_{m+1}-1}\,as\ z\to 1$ which shows that the bound in (6.3) is best possible.

Convolution Properties

This section concentrates on a way to derive the convolution properties by using Schild and Silverman (1975) techniques. At the beginning, let's recall the following definition: Let the functions $f_i(z)(i=1,2)$ be defined by:

$$f_i(z)=z^{-p}-\sum_{n=p}^{\infty}a_{n,i}z^n\left(a_n\geq 0;p\in\mathbb{N};i=1,2\right)\quad(7.1)$$

The modified Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is defined by:

$$(f_1*f_2)(z)=z^{-p}-\sum_{n=p}^{\infty}a_{n,i}z^n\quad(7.2)$$

Theorem 6: Let the functions $f_1(z)$ $(i=1,2)$ defined by (7.1) be in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$ and $b_n\geq b_p\geq 1(n\geq p)$. Then $(f_1*f_2)(z)\in\mathcal{S}_g^*(A,B,\gamma,p,j)$, where $\gamma=p+j-\frac{(p+j-\alpha)^2(B-A)(2p)\frac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\left[(1+B)(2p)+(p+j-\alpha)(B-A)\right]^2 b_p-(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$.

The result is sharp for the functions $f_i(z)$ $(i=1,2)$ given by:

$$f_i(z)=z^{-p}-\frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\begin{bmatrix}(1+B)2p\\+(p+j-\alpha)(B-A)\end{bmatrix}b_p}z^p\quad(7.3)$$
$$\left(i=1,2;p\in N;j\in 2\mathbb{N}U\{0\};z\in U^*\right)$$

Proof. In view of Theorem 1, it suffices to prove that:

$$\sum_{n=p}^{\infty} \frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\gamma)(B-A)\right]}{(p+j-\gamma)(B-A)\frac{(p+j-1)!}{(p-1)!}} a_{n,1}a_{n,2}b_n \le 1, \left(p \in N; j \in 2\mathbb{N}U\{0\}\right)$$

Since $f_i(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)(i=1,2)$, we have:

$$\sum_{n=p}^{\infty} \frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} a_{n,i}b_n \le 1, \left(i=1,2; p \in N; j \in 2\mathbb{N}U\{0\}\right)$$

Therefore, by the Cauchy-Schwarz inequality, we obtain:

$$\sum_{n=p}^{\infty} \frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} \sqrt{a_{n,1}a_{n,2}}b_n \le 1 \left(p \in N; j \in 2\mathbb{N}U\{0\}\right) \quad (7.4)$$

Hence, we need to show that

$$\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\gamma)(B-A)\right]}{(p+j-\gamma)(B-A)\frac{(p+j-1)!}{(p-1)!}} a_{n,1}a_{n,2}b_n \le \frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}} \sqrt{a_{n,1}a_{n,2}}b_n, \left(n \ge p; p \in N; j \in 2\mathbb{N}\mathbb{U}\{0\}\right)$$

Or, equivalently, that:

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right](p+j-\gamma)}{\left[(1+B)(n+p)+(p+j-\gamma)(B-A)\right](p+j-\alpha)} \left(n \ge p; p \in N; j \in 2\mathbb{N}\mathbb{U}\{0\}\right)$$

From the equality (7.4), we have

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]b_n}, \left(n \ge p; p \in N; j \in 2\mathbb{N}\mathbb{U}\{0\}\right)$$

So, it is sufficient to prove that:

$$\frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]b_n} \le \frac{\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right](p+j-\gamma)}{\left[(1+B)(n+p)+(p+j-\gamma)(B-A)\right](p+j-\alpha)}, \quad (7.5)$$
$$\left(n \ge p; p \in N; j \in 2\mathbb{N}\mathbb{U}\{0\}\right)$$

It is follows from (7.5) that:

$$\gamma \le p+j-\frac{(p+j-\alpha)^2(B-A)(n+p)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]^2 b_n-(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$

Now, defining the function φ(*n*) by:

$$\varphi(n)=p+j-\frac{(p+j-\alpha)^2(B-A)(n+p)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]^2 b_n-(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$

We easily see that φ(*n*) is an increasing function of *n*. So we have:

$$\gamma \le \varphi(p)=p+j-\frac{(p+j-\alpha)^2(B-A)(2p)\frac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\left[(1+B)(2p)+(p+j-\alpha)(B-A)\right]^2 b_p-(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$

Which completes the proof of Theorem 6.

Theorem 7: Let the function $f_1(z)$ defined by (7.1) be in the class $\mathcal{S}_g^*(A,B,\varepsilon,p,j)$ and the function $f_2(z)$ defined by (7.1) be in the class $\mathcal{S}_g^*(A,B,\varepsilon,p,j)$ provided $b_n \ge b_p \ge 1 (n \ge p)$. Then $(f_1 * f_2)(z) \in \mathcal{S}_g^*(A,B,\theta,p,j)$, where:

$$\theta = p+j-\frac{(p+j-\alpha)(p+j-\varepsilon)(B-A)(2p)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\mu(\alpha)\mu(\varepsilon)b_p-(p+j-\alpha)(p+j-\varepsilon)(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$

Where:

$$\mu(\rho)=\left[(1+B)(2p)+(p+j-\rho)(B-A)\right]$$

The result is sharp for the functions $f_i(z)(i = 1,2)$ given b:

$$f_1(z) = z^{-p} - \frac{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\left[(1+B)2p+(p+j-\alpha)(B-A)\right]b_p} z^p$$
$$\left(n \ge p; p \in N; j \in 2\mathbb{N}\cup\{0\}; z \in U^*\right)$$

$$f_2(z) = z^{-p} - \frac{(p+j-\varepsilon)(B-A)\frac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\left[(1+B)2p+(p+j-\varepsilon)(B-A)\right]b_p} z^p$$
$$\left(n \ge p; p \in N; j \in 2\mathbb{N}\cup\{0\}; z \in U^*\right)$$

Theorem 8: Let the functions $f_i(z)(i = 1,2)$ defined by (7.1) be in the class $\mathcal{S}_g^*(A,B,\varepsilon,p,j)$ and $b_n \ge b_p \ge 1 (n \ge p)$. Then the function $h(z)$ defined by:

$$h(z) = z^{-p} - \sum_{n=p}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n$$

Belongs to the class $\mathcal{S}_g^*(A,B,\varepsilon,p,j)$, where:

$$\omega = p + j - \frac{2(p+j-\alpha)^2(B-A)(2p)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(2p)+(p+j-\alpha)(B-A)\right]^2 b_n - 2(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$

The result is sharp for the functions $f_i(z)(i = 1,2)$ given by (7.3).

Proof. Since $f_i(z) \in \mathcal{S}_g^*(A,B,\alpha,p,j)(i=1,2)$. Thus, by Theorem 1:

$$\left[\sum_{n=p}^{\infty}\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}a_{n,i}b_n\right]^2$$
$$\le 1, (i = 1,2; p \in N; j \in 2\mathbb{N}\cup\{0\})$$

Note that:

$$\sum_{n=p}^{\infty}\left[\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}\right]^2 a_{n,i}^2 b_n^2$$
$$\le \left[\sum_{n=p}^{\infty}\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}a_{n,i}b_n\right]^2 \le 1 \quad (7.6)$$
$$\left(i = 1,2; p \in N; j \in 2\mathbb{N}\cup\{0\}\right)$$

Implies that:

$$\sum_{n=p}^{\infty}\frac{1}{2}\left[\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}\right]^2$$
$$(a_{n,1}^2 + a_{n,2}^2)b_n^2 \le 1, \ \left(i = 1,2; p \in N; j \in 2\mathbb{N}\cup\{0\}\right)$$

So, we need to find the largest ω such that:

$$\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\omega)(B-A)\right]}{(p+j-\omega)(B-A)\frac{(p+j-1)!}{(p-1)!}}b_n$$
$$\le \frac{1}{2}\left[\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\frac{(p+j-1)!}{(p-1)!}}b_n\right]^2,$$
$$\left(n \ge p; p \in N; j \in 2\mathbb{N}\cup\{0\}\right)$$

That is:

$$\omega \le\le p + j \frac{2(p+j-\alpha)^2(B-A)(n+p)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]^2 b_n - 2(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$
$$\left(n \ge p; p \in N; j \in 2\mathbb{N}\cup\{0\}\right)$$

Now, defining the function υ(*n*) by:

$$\nu(n) = p + j - \frac{2(p+j-\alpha)^2(B-A)(n+p)\frac{(p+j-1)!}{(p-1)!}}{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]^2 b_n - 2(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$
$$\left(n \ge p; p \in N; j \in 2\mathbb{N}\cup\{0\}\right)$$

It is clear that φ(*n*) is an increasing function of *n*. Thus, we have:

$$\gamma \le \nu(p) = p + j - \frac{2(p+j-\alpha)^2(B-A)(2p)\frac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\left[(1+B)(2p)+(p+j-\alpha)(B-A)\right]^2 b_p - 2(p+j-\alpha)^2(B-A)^2\frac{(p+j-1)!}{(p-1)!}}$$

Which complete the proof of Theorem 9.

Integral Operator

By using the techniques presented in (Srivastava *et al.*, 1998; Aouf, 2010) the below theorem shows the obtained integral transforms of functions in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$.

Theorem 9: If $f(z)\in\Sigma^*(p)$ is in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$ and $b_n \geq b_p \geq 1 (n \geq p)$, then the integral transforms:

$$F_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)\,du,\left(0<u\leq 1;0<c<\infty\right) \quad (8.1)$$

Are in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$, where:

$$\vartheta = p+j-\frac{c(p+j-\alpha)(1+B)(2p)}{(c+2p)\left[(1+B)(2p)+(p+j-\alpha)(B-A)\right]-c(p+j-\alpha)(B-A)}$$

The result is the best possible for the function $f(z)$ given by:

$$f(z)=z^{-p}-\frac{(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}{\delta(p,j)\left[(1+B)(2p)+(p+j-\alpha)(B-A)\right]b_p}z^p,$$
$$\left(p\in N; j\in 2\mathbb{N}\cup\{0\}\right)$$

Proof. Let $f(z)=z^{-p}-\sum_{n=p}^{\infty}a_n z^n$ in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$. Then:

$$F_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)\,du=z^{-p}-\sum_{n=p}^{\infty}\frac{c}{c+p+n}a_n z^n$$

We need to find the largest ϑ such that:

$$\sum_{n=p}^{\infty}\frac{c\,\delta(n,j)\left[(1+B)(n+p)+(p+j-\vartheta)(B-A)\right]}{(c+p+n)(p+j-\vartheta)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}a_n b_n\leq 1 \quad (8.2)$$

Since $f(z)\in\mathcal{S}_g^*(A,B,\alpha,p,j)$ and by Theorem 1, we get:

$$\sum_{n=p}^{\infty}\frac{\delta(n,j)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)(B-A)\dfrac{(p+j-1)!}{(p-1)!}}a_n b_n\leq 1$$

Note that the inequality (8.2) is satisfied if:

$$\frac{c\left[(1+B)(n+p)+(p+j-\vartheta)(B-A)\right]}{(c+p+n)(p+j-\vartheta)}\leq\frac{\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]}{(p+j-\alpha)}$$

Or equivalently:

$$\vartheta\leq p+j-\frac{c(p+j-\alpha)(1+B)(n+p)}{(c+p+n)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]-c(p+j-\alpha)(B-A)} \quad (8.3)$$

Let:

$$F(n)=p+j-\frac{c(p+j-\alpha)(1+B)(n+p)}{(c+p+n)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]-c(p+j-\alpha)(B-A)}$$

Then $F_{c+p-1}(n)$ is an increasing function of n. Since:

$$F(p)=p+j-\frac{c(p+j-\alpha)(1+B)(2p)}{(c+p+n)\left[(1+B)(2p)+(p+j-\alpha)(B-A)\right]-c(p+j-\alpha)(B-A)}$$

This completes the proof.

Integral Representation

In the following theorem, we determine the integral representation of functions that belong to the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$.

Theorem 10: Let $f(z)\in\Sigma^*(p)$ be in the class $\mathcal{S}_g^*(A,B,\alpha,p,j)$. Then:

$$(f*g)^{(j)}(z)=e^{\int_0^z\frac{\varphi(z)\left[(1+B)(n+p)+(p+j-\alpha)(B-A)\right]-(p+j)}{z(1-\varphi(z)B}}$$

where, $|\varphi(t)|<1$, $z\in U^*$:

Proof. By putting $\frac{z(f*g)^{(j+1)}(z)}{(f*g)^{(j)}(z)}=M(z)$ in (1.4), we have:

$$\left|\frac{M(z)+(p+j)}{BM(z)+\left[B(p+j)+(p+j-\alpha)(B-A)\right]}\right|<1$$

Or equivalently:

$$\frac{M(z)+(p+j)}{BM(z)+\left[B(p+j)+(p+j-\alpha)(B-A)\right]}=\varphi(t),|\varphi(t)|<1,z\in U^*$$

So:

$$\frac{(f*g)^{(j+1)}(z)}{(f*g)^{(j)}(z)}=\frac{\varphi(z)\left[B(p+j)+(p+j-\alpha)(B-A)\right]-(p+j)}{z(1-\varphi(z)B)}$$

After integration, we get:

$$\log\left(\frac{(f*g)^{(j+1)}(z)}{(f*g)^{(j)}(z)}\right)=\int_0^z\frac{\varphi(z)\left[B(p+j)+(p+j-\alpha)(B-A)\right]-(p+j)}{z(1-\varphi(z)B)}$$

Thus:

$$\frac{(f*g)^{(j+1)}(z)}{(f*g)^{(j)}(z)}=e^{\int_0^z\frac{\varphi(z)\left[B(p+j)+(p+j-\alpha)(B-A)\right]-(p+j)}{z(1-\varphi(z)B)}}$$

This completes the proof.

Conclusion

The purpose of this study was to introduce a certain new subclass of class of meromorphically multivalent functions with negative coefficients. We also investigated the geometric characterization of the coefficient estimates for functions in this subclass. Moreover, we obtained the distortion theorem, radii of meromorphically p-valent starlikeness and convexity, neighborhood property, partial sums, convolution properties, integral operator and integral representation.

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Author's Contributions

All authors jointly worked on deriving the results and approved the final manuscript.

Ethics

The authors declare no conflict of interest.

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