Original Research Paper

Explicit Direct Integrators of RK Type for Solving Special Fifth-Order Ordinary Differential Equations

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Abstract: The main contribution of this paper is the development of direct explicit methods of Runge-Kutta (RK) type for solving special fifth-Order Ordinary Differential Equations (ODEs). For this purpose, we have generalized RKD and RKT methods of special third and fourth-order ODEs. Using Taylor expansion, we have derived the algebraic equations of algebraic equations of order conditions for the proposed RKM integrators up to the eighth order. Based on these conditions, two RKM methods of orders five and six with three and four-stage are derived. Numerical implementation shows that the new methods agree well with existing RK methods, but requires less function evaluations. This is so due to the fact that RKM methods are direct; hence, they save considerable amount of computational time.

Keywords: RK, RKN, RKD, RKT and RKFD Integrators, Runge-Kutta Methods, Special Fifth-Order, Four-Stage, Ordinary Differential Equations, Order Conditions, Taylor Series, Rooted Trees

Introduction

The mathematical modeling of many real-life problems in physics, engineering and economics can be written as higher-order differential equations, ordinary or partial, (DEs) model. Typical examples can be found in different fields such as quantum physics, solid state physics and fluid physics plasma physics Wazwaz (2006). For example, the nonlinear Korteweg-deVries (KdV), the nonlinear Schrodinger and the Kadomtsev-Petviashvili equations are important mathematical models in quantum mechanics and nonlinear optics. Fifth-order KdV equation has found applications in some branches of physics, such as capillary-gravity water waves, chains of coupled oscillators and magneto-acoustic waves in plasma Isaza *et al*. (2015; Lee, 2014; Wazwaz, 2006). Another application of fifth-order DEs is the nonlinear dispersive equations, which include several models arising in the study of different physical phenomena Isaza *et al*. (2015). A mathematical model describing weakly nonlinear long internal waves at the interface between two thin layers of different density, called the Gardner-Kawahara equation, which is a generalization of the famous KdV equation, contains a cubic term together a fifth-order dispersion term Lee (2014).

In this study, a derivation for nonlinear equations (order conditions) of direct explicit RKM methods for solving a special fifth-order ODEs is presented, along with two numerical methods for ODEs with orders five and six.

Special Fifth-Order Ordinary Differential Equations

In this study, we concerned with fifth-order ordinary differential equation with no appearance for the first, second, third and fourth derivatives $w^{(i)}(x)$, for $i = 1,2,3,4$. It can be written in the following form:

$$w^{(5)}(x) = g(x, w(x)), \ x \geq x_0 \tag{1}$$

Subject to initial condition:

$$w^{(i)}(x_0) = \gamma^i$$

for $i = 0,1,...,4$.
where:

$$g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

and:

$$w(x) = \left[w_1(x), w_2(x),..., w_N(x) \right]$$

for $i = 0,1,...,4$.

Knowing that, N is the components number of the vector of independent variables of the system of ordinary differential Equation 1. To convert the function $g(x, w(x))$ which depends on two variables, to a function which depends only on one variable $w(x)$, using high dimension we can work in $N+1$ dimension using the assumption $w_{N+1}(x) = x$, then (1) can be simplified to following Equation 2:

$$v^{(5)}(x) = h(v(x)) \qquad (2)$$

Using the following consideration:

$$v(x) = \begin{pmatrix} w_1(x) \\ w_2(x) \\ w_3(x) \\ \cdots \\ \cdots \\ \cdots \\ w_N(x) \\ x \end{pmatrix}, h(v) = \begin{pmatrix} g_1(v_1, v_2, \ldots, v_N, v_{N+1}) \\ g_2(v_1, v_2, \ldots, v_N, v_{N+1}) \\ g_3(v_1, v_2, \ldots, v_N, v_{N+1}) \\ \cdots \\ \cdots \\ \cdots \\ g_N(v_1, v_2, \ldots, v_N, v_{N+1}) \\ 0 \end{pmatrix}$$

Subject to the initial condition:

$$w^{(i)}(x_0) = \overline{\gamma^i}$$

for $i = 0, 1, \ldots, 4$.
where:

$$\overline{\gamma^i} = \left[\gamma_1^i, \gamma_2^i, \ldots, \gamma_N^i, x_0 \right]$$

for $i = 0, 1, \ldots, 4$.

These class of ODEs are found in many engineering and physical problems. Some of scientists and engineers can solve the Equation 1 or 2 using one of multistep methods. Almost, they used to solve higher-order ODE by converting it to equivalent system of first-order ODEs and can solve using a classical RK method Faires and Burden (2003). However, it would be more efficient if ODEs of order five can be solved using the proposed direct RKM method. The proposed method solve Equation 1 or 2 directly be more efficient since it has less function evaluations and computational time in implementation.

For review of RK type methods, the second-order ODEs, Sommeijer (1993 Van der Houwen and Sommeijer, 1989) have derived direct numerical integrators with constant step size while Cong (2001) has derived direct numerical methods with variable step-size for solving second-order ODEs while for third-order, Mechee *et al.* (2013b; You and Chen, 2013) have

derived direct integrators of RK type for solving ODEs of third-order while Senu *et al.* (2014) have derived variable step-size direct integrators of RK type of orders 6(5), 5(4) and 4(3) for solving third-order ODEs. Hussain *et al.* (2016) have derived direct numerical integrators with variable and constant step-size for solving fourth-order ODEs, moreover different orders of direct explicit RKD methods for solving special third-order ODEs with constant step-size have been derived (Mechee *et al.*, 2014a; 2014b; 2014c). However, the regions of stability for RKD methods have been derived by Mechee *et al.* (2016).

In this study, we are concerned with the one-step RKM integrators for directly solving fifth-order ODEs. To obtain the order conditions we used the Taylor series expansion approach. Consequently, we have derived two of direct RKM integrators based on the algebraic equations of order conditions of RKM integrators.

Proposed RKM Methods

The proposed formula of explicit RKM integrator with s-stage for solving fifth-order ODEs (1) can be written as follow:

$$w_{n+1} = w_n + w_n' + \frac{h^2}{2!}w_n'' + \frac{h^3}{3!}w_n''' + \frac{h^4}{4!}w_n'''' + h^5 \sum_{i=1}^{s} b_i k_i \qquad (3)$$

$$w_{n+1}' = w_n' + hw_n'' + \frac{h^2}{2!}w_n''' + \frac{h^3}{3!}w_n'''' + h^4 \sum_{i=1}^{s} b_i' k_i \qquad (4)$$

$$w_{n+1}'' = w_n'' + hw_n''' + \frac{h^2}{2!}w_n'''' + h^3 \sum_{i=1}^{s} b_i'' k_i \qquad (5)$$

$$w_{n+1}''' = w_n''' + hw_n'''' + h^2 \sum_{i=1}^{s} b_i''' k_i \qquad (6)$$

$$w_{n+1}'''' = w_n'''' + h \sum_{i=1}^{s} b_i'''' k_i \qquad (7)$$

where:

$$k_1 = g(x_n, w_n) \qquad (8)$$

$$k_i = g \begin{pmatrix} x_n + c_i h, w_n + hc_i w_n' + \frac{h^2}{2!}c_i^2 w_n'' \\ + \frac{h^3}{3!} + c_i^3 w_n''' \quad \frac{h^4}{4!}c_i w_{0n}'''' + h^5 \sum_{i=1}^{i-1} a_{ij} k_j \end{pmatrix} \qquad (9)$$

for $i = 2, 3, \ldots, s$. and h is the step-size. The parameters of RKM integrator are $a_{ij}, c_i, b_i'''', b_i''', b_i'', b_i'$ and b_i for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, s$ are real. It is an explicit integrator if $a_{ij} = 0$ for $i \leq j$ and otherwise its implicit integrator. We have expressed the coefficients RKM method in Butcher Table as follows:

$$\frac{c\,|\,A}{\begin{array}{c} |\,b^T \\ |\,b'^T \\ |\,b''^T \\ |\,b'''^T \\ |\,b''''^T \end{array}}$$

The order conditions of RKD integrators for solving third-order ODEs have been derived by Mechee *et al.* (2013a; You and Chen, 2013) while the order conditions of RKFD integrators for solving fourth-order ODEs have been derived by Hussain *et al.* (2015). In this study, using the same technique, we have derived the algebraic equations of order conditions of RKM methods for solving special fifth-order ODEs.

The Order Conditions Derivation of RKM Methods

This approach has a special characteristic history, it deals with non-scalar problems in the hope that this generalizes correctly to high dimensions. It is similar to those which were used by Runge, Heun, Kutta, Nystrom, Huta and others to obtain methods up to order six for solving the first-order problem $y' = f(x, y)$ Butcher and Wanner (1996; Butcher, 2008). The algebraic equations of order conditions of RKM integrators can be obtain from the direct expansion of the local truncation error. This idea is based on the derivation of algebraic equations of order conditions for RK methods introduced in (Dormand, 1996). The RKM formulae in (3-7) can be expressed as follow:

$$
\begin{aligned}
w_{n+1} &= w_n + h\Phi\left(x_n, w_n\right), \\
w'_{n+1} &= w'_n + h\Phi'\left(x_n, w_n\right), \\
w''_{n+1} &= w''_n + h\Phi''\left(x_n, w_n\right), \\
w'''_{n+1} &= w'''_n + h\Phi'''\left(x_n, w_n\right), \\
w''''_{n+1} &= w''''_n + h\Phi''''\left(x_n, w_n\right)
\end{aligned}
$$

where, the increment functions are defined as the following:

$$
\begin{aligned}
\Phi\left(x_n, w_n\right) &= w'_n + \frac{h}{2!}w''_n + \frac{h^2}{3!}w'''_n + \frac{h^3}{4!}w''''_n + h^4\sum_{i=1}^{s} b_i k_i, \\
\Phi'\left(x_n, w_n\right) &= w''_n + \frac{h}{2!}w'''_n + \frac{h^2}{3!}w''''_n + h^3\sum_{i=1}^{s} b'_i k_i, \\
\Phi''\left(x_n, w_n\right) &w'''_n + \frac{h}{2!}w''''_n + h^2\sum_{i=1}^{s} b''_i k_i, \\
\Phi'''\left(x_n, w_n\right) &w''''_n + h\sum_{i=1}^{s} b'''_i k_i, \\
\Phi''''\left(x_n, w_n\right) &= \sum_{i=1}^{s} b''''_i k_i
\end{aligned}
$$

where, k_i is defined in the formula (8) and (9). If D represents Taylor series increment function and the local truncation errors of the derivatives of the solution of order zero up to the fourth-order cab be obtained by substituting the exact solution $w(x)$ of ODE (1) into the RKM increment function. This gives:

$$t_{n+1}^{(i)} = h\left(\Phi^{(i)} - \Delta^{(i)}\right)$$

for $i = 0,1,...,4$.

These expressions have given in elementary differentials terms also Taylor series increment can be expressed as follows:

$$
\begin{aligned}
\Delta^{(0)} &= w' + \frac{h}{2!}w'' + \frac{h^2}{3!}w^{(3)} + \frac{h^3}{4!}w^{(4)} + O\left(h^4\right), \\
\Delta^{(1)} &= w'' + \frac{h}{2!}w^{(3)} + \frac{h^2}{3!}w^{(4)} + O\left(h^3\right), \\
\Delta^{(2)} &= w^{(3)} + \frac{h}{2!}w^{(4)} + O\left(h^2\right), \\
\Delta^{(3)} &= w^{(4)} + O\left(h\right), \\
\Delta^{(4)} &= O\left(1\right)
\end{aligned}
$$

Hence, for the scalar case function, the few first differentials of the function g are given as follow:

$$
\begin{aligned}
G_1^{(5)} &= g, \\
G_1^{(6)} &= g_x + g_w w', \\
G_1^{(7)} &= g_{xx} + 2g_{xw}w' + g_w w'' + g_{ww}\left(w'\right)^2, \\
G_1^{(8)} &= g_{xxx} + w'^3 g_{www} + g_{xww}\left(2w' + w'^2\right) \\
&\quad + 3w'w''g_{ww} + 3w''g_{xw} + 3w'g_{xwx} + w'''g_w, \\
G_1^{(9)} &= g_{xxxx} + w'g_{xxxw} + \left(w'\right)^3\left(w'g_{wwww} + g_{wwwx}\right) \\
&\quad + 3\left(w'\right)^2 w''g_{www} + 2w''g_{xww}\left(1 + w'\right) + w', \\
&\quad \left(g_{xwwx} + y'g_{xwww}\left(2 + w'\right) + 3w'w''\left(w'g_{www} + g_{wwx}\right)\right. \\
&\quad + 3g_{ww}\left(w' + w''\left(w'' + w'\right)\right) + 3w'''g_{xw} \\
&\quad + 3w''\left(g_{xwx} + w'g_{xww}\right) + 3w''g_{xwx} + 3w'\left(g_{xwxx} + \right)w'g_{xwxw}\right) \\
&\quad + w'''g_w + w''\left(g_{wx} + w'g_{ww}\right)
\end{aligned}
$$

The increment functions $\Phi^{(0)}$, $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$ and $\Phi^{(4)}$ for the RKM integrators can be written as follow using the above terms:

$$
\begin{aligned}
\sum_{i=1}^{s} b_i k_i &= \sum_{i=1}^{s} b_i g + \sum_{i=1}^{s} b_i c_i\left(g_x + g_y y'\right)h + \\
&\quad \frac{1}{2}\sum_{i=1}^{s} b_i c_i^2\left(g_{xx} + 2g_{xy}y' + g_y y'' + g_{yy}\left(y'\right)^2\right)h^2 + O\left(h^3\right)
\end{aligned}
\tag{10}
$$

$$\sum_{i=1}^{s} b'_i \, k_i = \sum_{i=1}^{s} b'_i \, g + \sum_{i=1}^{s} b'_i \, c_i \left(g_x + g_y y'\right)h +$$
$$\frac{1}{2}\sum_{i=1}^{s} b'_i \, c_i^2 \left(g_{xx} + 2g_{xy}y' + g_y y'' + g_{yy}\left(y'\right)^2\right)h^2 + O\left(h^3\right) \tag{11}$$

$$\sum_{i=1}^{s} b''_i \, k_i = \sum_{i=1}^{s} b''_i \, g + \sum_{i=1}^{s} b''_i \, c_i \left(g_x + g_y y'\right)h +$$
$$\frac{1}{2}\sum_{i=1}^{s} b''_i \, c_i^2 \left(g_{xx} + 2g_{xy}y' + g_y y'' + g_{yy}\left(y'\right)^2\right)h^2 + O\left(h^3\right) \tag{12}$$

$$\sum_{i=1}^{s} b'''_i \, k_i = \sum_{i=1}^{s} b'''_i \, g + \sum_{i=1}^{s} b'''_i \, c_i \left(g_x + g_y y'\right)h +$$
$$\frac{1}{2}\sum_{i=1}^{s} b'''_i \, c_i^2 \left(g_{xx} + 2g_{xy}y' + g_y y'' + g_{yy}\left(y'\right)^2\right)h^2 + O\left(h^3\right) \tag{13}$$

$$\sum_{i=1}^{s} b''''_i \, k_i = \sum_{i=1}^{s} b''''_i \, g + \sum_{i=1}^{s} b''''_i \, c_i \left(g_x + g_y y'\right)h +$$
$$\frac{1}{2}\sum_{i=1}^{s} b''''_i \, c_i^2 \left(g_{xx} + 2g_{xy}y' + g_y y'' + g_{yy}\left(y'\right)^2\right)h^2 + O\left(h^3\right) \tag{14}$$

The expressions for the local truncation errors in the derivatives of the solution up to the fourth-order, $y^{(i)}(x)$ for $i = 0,1,2,3,4$ are written as follow:

$$t_{n+1}^{(0)} = h^5 \left[\sum_{i=1}^{s} b_i \, k_i - \left(\frac{1}{120}G_1^{(5)} + \frac{1}{720}G_1^{(6)} + \frac{1}{5040}G_1^{(7)} + \frac{1}{40320}G_1^{(8)} + \ldots\right)\right] \tag{15}$$

$$t_{n+1}^{(1)} = h^4 \left[\sum_{i=1}^{s} b'_i \, k_i - \left(\frac{1}{24}G_1^{(5)} + \frac{1}{120}G_1^{(6)} + \frac{1}{720}G_1^{(7)} + \frac{1}{5040}G_1^{(8)} + \ldots\right)\right] \tag{16}$$

$$t_{n+1}^{(2)} = h^3 \left[\sum_{i=1}^{s} b''_i \, k_i - \left(\frac{1}{6}G_1^{(5)} + \frac{1}{24}G_1^{(6)} + \frac{1}{120}G_1^{(7)} + \frac{1}{720}G_1^{(8)} + \ldots\right)\right] \tag{17}$$

$$t_{n+1}^{(3)} = h^2 \left[\sum_{i=1}^{s} b'''_i \, k_i - \left(\frac{1}{2}G_1^{(5)} + \frac{1}{6}G_1^{(6)} + \frac{1}{24}G_1^{(7)} + \frac{1}{120}G_1^{(8)} + \ldots\right)\right] \tag{18}$$

$$t_{n+1}^{(4)} = h \left[\sum_{i=1}^{s} b''''_i \, k_i - \left(G_1^{(5)} + \frac{1}{2}G_1^{(6)} + \frac{1}{6}G_1^{(7)} + \frac{1}{24}G_1^{(8)} + \frac{1}{120}G_1^{(9)} + \ldots\right)\right] \tag{19}$$

Substitute the Equations (10-14) into the Equations (15-19) respectively and expanding as a Taylor expansion. Using computer algebra which has written by MAPLE programming (Gander and Gruntz, 1999) then, the error equations or the algebraic equations of order conditions up to order-seven can be expressed in rooted trees form (Table 1-5) which have all indices are from 1 to s for $w^{(i)}$ for $i = 0,1,\ldots,4$.

Derivation of RKM Methods

To derive RKM methods of orders fifth and sixth, we have used the algebraic conditions up to order of the method for $w^{(i)}$ for $i = 0,1,\ldots,4$ in Table 1-5, respectively, with the following assumption:

$$b'''_i = \left(1 - c_i\right)b''''_i, \quad b''_i = \frac{\left(1 - c_i\right)^2}{2!}b''''_i,$$
$$b'_i = \frac{\left(1 - c_i\right)^3}{3!}b''''_i, \quad b_i = \frac{\left(1 - c_i\right)^4}{4!}b''''_i$$

for $i = 1,\ldots,s$, which imposed to reduce the number of algabraic order equations to be solve. The parameters of RKM method are $c_i, a_{ij}, b_i, b'_i, b''_i, b'''_i, b''''_i$ for $i = 1,2,\ldots,s$ and $j = 1,2,\ldots,s$. The parameters of the new methods have been evaluated by solving the system of algebraic equations of order conditions in (1-5).

Three-stage fifth-order and four-stage sixth-order direct RKM integrators have been derived and the Butcher tableaus of these integrators are shown in the Table 6 and 7, respectively.

Table 1. Rooted trees of order conditions for w

$|t|$	t	$S(t)$	$\phi(t)$	$\gamma(t)$
5			$\sum_{i=1}^{s} b_i$	120
6			$\sum_{i=1}^{s} b_i c_i$	720
7			$\sum_{i=1}^{s} b_i c_i^2$	2520
7			$\sum_{i=1}^{s} b_i c_i^3$	840

Table 2. Rooted trees of order conditions for w'

$\lvert t \rvert$	t	$S(t)$	$\phi(t)$	$\gamma(t)$
4			$\sum_{i=1}^{s} b_i'$	24
5			$\sum_{i=1}^{s} b_i' c_i$	120
6			$\sum_{i=1}^{s} b_i' c_i^2$	630
7			$\sum_{i=1}^{s} b_i' c_i^3$	840
8			$\sum_{i=1}^{s} b_i' a_{ij}$	5040
8			$\sum_{i=1}^{s} b_i' c_i^4$	210

Table 3. Rooted trees of order conditions for w''

$\lvert t \rvert$	t	$S(t)$	$\phi(t)$	$\gamma(t)$
3			$\sum_{i=1}^{s} b_i''$	6
4			$\sum_{i=1}^{s} b_i'' c_i$	24
5			$\sum_{i=1}^{s} b_i'' c_i^2$	60
6			$\sum_{i=1}^{s} b_i'' c_i^3$	120
7			$\sum_{i=1}^{s} b_i'' a_{ij}$	720
7			$\sum_{i=1}^{s} b_i'' c_i^4$	210
8			$\sum_{i=1}^{s} b_i'' a_{ij} c_j$	5040

Table 4. Rooted trees of order conditions for w'''

$\lvert t \rvert$	t	$S(t)$	$\phi(t)$	$\gamma(t)$
2			$\sum_{i=1}^{s} b_i'''$	2
3			$\sum_{i=1}^{s} b_i''' c_i$	6
4			$\sum_{i=1}^{s} b_i''' c_i^2$	12
5			$\sum_{i=1}^{s} b_i''' c_i^3$	20
6			$\sum_{i=1}^{s} b_i''' c_i^4$	30
7			$\sum_{i=1}^{s} b_i'' a_{ij} c_j$	5040
7			$\sum_{i=1}^{s} b_i''' c_i^5$	42

Table 5. Rooted trees of order conditions for w''''

$|t|$	t	$S(t)$	$\phi(t)$	$\gamma(t)$
1			$\sum_{i=1}^{s} b_i''''$	1
2			$\sum_{i=1}^{s} b_i'''' c_i$	2
3			$\sum_{i=1}^{s} b_i'''' c_i^2$	3
4			$\sum_{i=1}^{s} b_i'''' c_i^3$	4
5			$\sum_{i=1}^{s} b_i'''' c_i^4$	5
6			$\sum_{i=1}^{s} b_i'''' c_i^5$	6
6			$\sum_{i=1}^{s} b_i'''' a_{ij}$	720
7			$\sum_{i=1}^{s} b_i'''' a_{ij} c_j$	5040
7			$\sum_{i=1}^{s} b_i'''' c_i a_{ij}$	840
7			$\sum_{i=1}^{s} b_i'''' c_i^6$	7

Table 6. The butcher tableau RKM5 Method

$$
\begin{array}{c|ccc}
0 & 0 & & \\
\dfrac{3}{5}-\dfrac{\sqrt{6}}{10} & \dfrac{1}{2} & 0 & \\
\dfrac{3}{5}+\dfrac{\sqrt{6}}{10} & \dfrac{1}{2} & \dfrac{1}{2} & 0 \\
\hline
 & 1 & 0 & -\dfrac{119}{120} \\
 & -\dfrac{1}{40}-\dfrac{\sqrt{6}}{360} & \dfrac{1}{60}+\dfrac{\sqrt{6}}{360} & 0 \\
 & \dfrac{1}{18} & \dfrac{1}{18}-\dfrac{\sqrt{6}}{48} & \dfrac{1}{18}+\dfrac{\sqrt{6}}{48} \\
 & \dfrac{1}{9} & \dfrac{7}{36}-\dfrac{\sqrt{6}}{48} & \dfrac{7}{18}-\dfrac{\sqrt{6}}{18} \\
 & \dfrac{1}{9} & \dfrac{7}{36}-\dfrac{\sqrt{6}}{48} & \dfrac{7}{18}-\dfrac{\sqrt{6}}{18}
\end{array}
$$

Table 7. The butcher tableau RKM6 Method

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
\dfrac{1}{2} & \dfrac{1}{2} & 0 & & \\
\dfrac{1}{2}+\dfrac{\sqrt{15}}{10} & 0 & \dfrac{1}{2} & 0 & \\
\dfrac{1}{2}-\dfrac{\sqrt{15}}{10} & 0 & \dfrac{1}{2} & -\dfrac{350}{200} & 0 \\
\hline
 & \dfrac{1}{180} & -\dfrac{359}{360} & \dfrac{1}{2} & \dfrac{1}{2} \\
 & -\dfrac{67}{4}-\dfrac{109\sqrt{15}}{360} & \dfrac{-59}{60}-\dfrac{109\sqrt{540}}{360} & \dfrac{1}{2} & \dfrac{145}{72}+\dfrac{109\sqrt{15}}{216} \\
 & 0 & \dfrac{1}{18} & \dfrac{1}{18}-\dfrac{\sqrt{15}}{72} & \dfrac{1}{18}+\dfrac{\sqrt{15}}{72} \\
 & 0 & \dfrac{2}{9} & \dfrac{5}{36}-\dfrac{\sqrt{15}}{36} & \dfrac{5}{36}-\dfrac{\sqrt{15}}{36} \\
 & 0 & \dfrac{4}{9} & \dfrac{5}{18} & \dfrac{5}{18}
\end{array}
$$

Numerical Implementation

In this section, a set of fifth-order ODEs is solved by using the fifth and sixth –order RKM methods. Then, these problems of fifth-order ODEs are reduced to a first-order ODEs systems and solved by using existing RK integrators of the same order, five and six

respectively. The numerical results are compared in Figure 2-4 to indicate the log of maximum absolute errors against the log of total time. The notations that were used are as follows:

- Step: Stepsize used
- IC: Initial condition
- Ex: Exact solution
- RKM5: The direct RKM method of fifth-order

- RKM6: The direct RKM method of sixth-order
- RK5: Existing RK method of fifth-order
- RK6: Existing RK method of sixth-order as given by Dormand (1996)
- Total Time/Time: The total time in seconds to solve the problems
- Max Error: Maximum $|y_{(xn)}-y_n|$ is maximum of absolute errors of the true solutions and the computed solutions

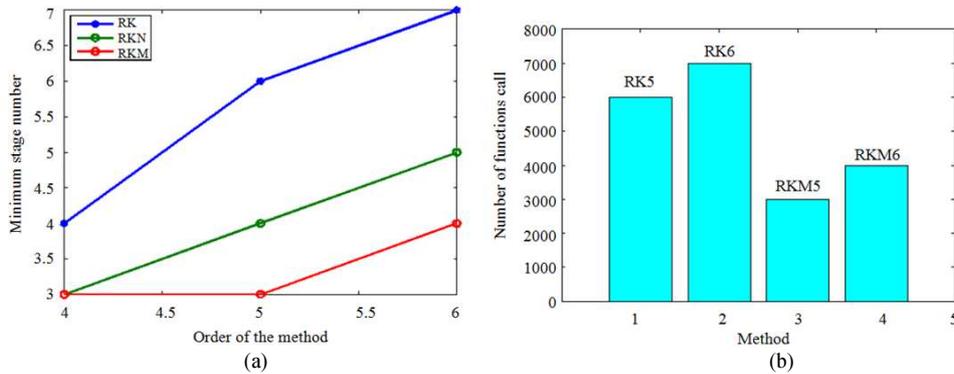(a)   (b)

Fig. 1. (a) Minimum stage number versus method order for RK, RKN and the recently proposed RKM method, (b) Number of function calls for RK and RKM methods with orders 5,6
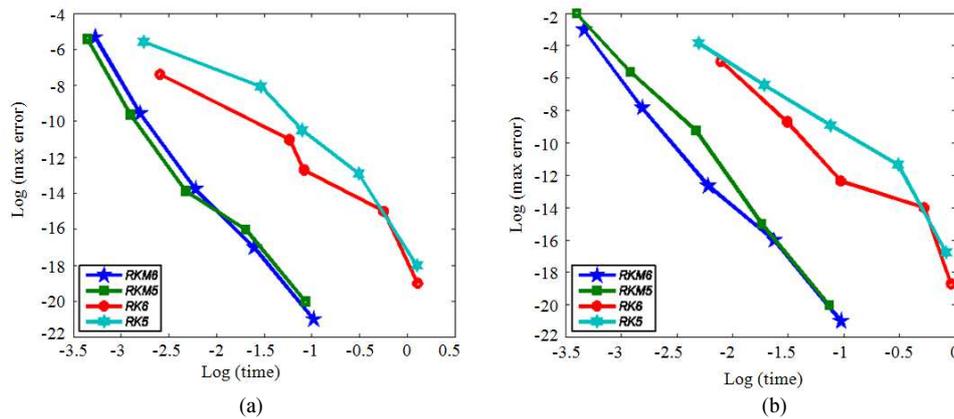

(a)   (b)

Fig. 2. Errors versus computational time for RKM5, RKM6, RK5 and RK6 methods in problems (a) 1 and (b) 2

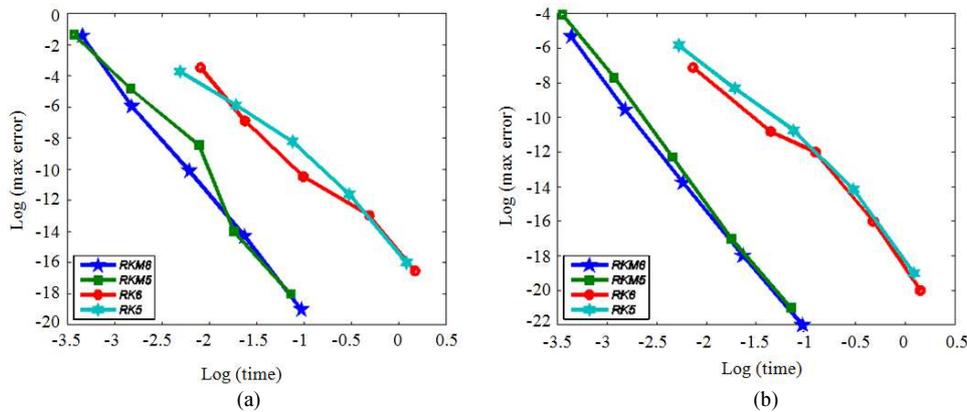(a)   (b)

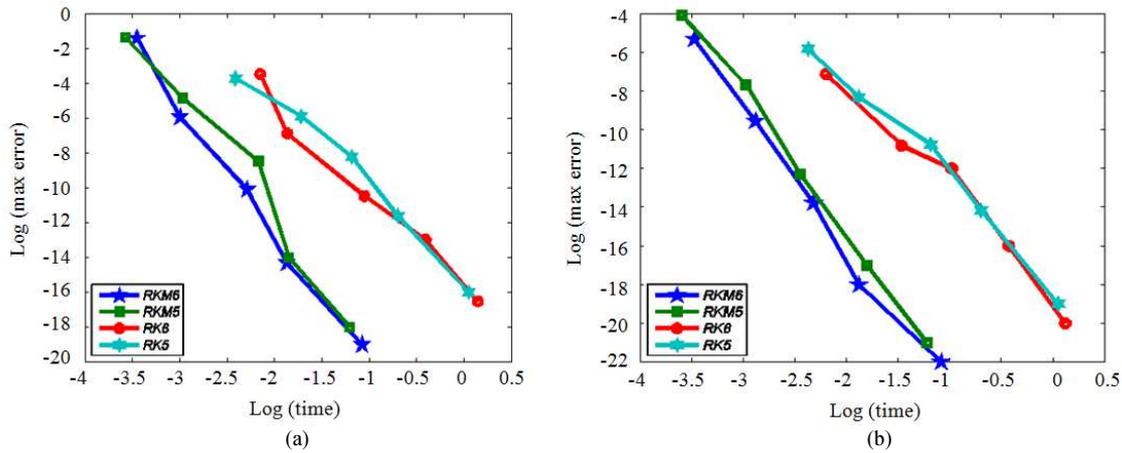Fig. 3. Errors versus computational time for RKM5, RKM6, RK5 and RK6 methods in problems (a) 3 and (b) 4

Fig. 4. Errors versus computational time for RKM5, RKM6, RK5 and RK6 methods in problems (a) 5 and (b) 6

Problems Tested of ODEs

Problem 1 (Linear ODE)

$$y^{(5)}(t) = \cos(t); \quad 0 < t \le \pi$$

- IC: $y^{(2i+1)}(0) = (-1)^i$, $y^{(2i)}(0) = 0$, for $i = 0,1$ and $y^{(4)}(0) = 0$
- Ex: $y(t) = \sin(t)$

Problem 2 (Non Constant Coefficients ODE)

$$y^{(5)}(t) = \left(-32t^5 + 16t^4 - 8t^3 + 4t^2 - 2t\right)y(t), \quad 0 < t \le 1$$

- IC: $y(0) = 1$, $y^{(i)}(0) = 0$, for $i = 1,2,3,4$
- Ex: $y(t) = e^{-t^2}$

Problem 3 (Non Linear ODE)

$$y^{(5)}(t) = -120y^6(t), \quad 0 < t \le 1$$

- IC: $y^{(i)}(0) = (-1)^i i!$ for $i = 0,1,...,4$
- Ex: $y(t) = \dfrac{1}{1+t}$

Problem 4 (Linear ODE)

$$y^{(5)}(t) = -y(t), \quad 0 < t \le 1$$

- IC: $y^{(i)}(0) = (-1)^i$ for $i = 0,1,...,4$
- Ex: $y(t) = e^{-t}$

Problem 5 (Linear ODE with Relatively Long Interval)

$$y^{(5)}(t) = 0.00001y(t), \quad 0 < t \le 10$$

- IC: $y^{(i)}(0) = (-0.1)^i$ for $i = 0,1,...,4$
- Ex: $y(t) = e^{-\frac{1}{10}}$

Problem 6 (Linear System of ODEs)

$$y_1^{(5)}(t) = -212y_1(t) - 180y_2(t) - 211y_3(t),$$
$$y_2^{(5)}(t) = -212y_1(t) - 179y_2(t) - 211y_3(t),$$
$$y_3^{(5)}(t) = -242y_1(t) - 242y_2(t) - 243y_3(t)$$

IC:

$$y_1(0) = 1, \ y_1'(0) = -2, \ y_1''(0) = 6, \ y_1'''(0) = -20, \ y_1^{(4)}(0) = 66$$
$$y_2(0) = 0, \ y_2'(0) = 1, \ y_2''(0) = -5, \ y_2'''(0) = 19, \ y_2^{(4)}(0) = 65$$
$$y_3(0) = 0, \ y_3'(0) = -2, \ y_3''(0) = 8, \ y_3'''(0) = -26, \ y_3^{(4)}(0) = 80$$

The system is integrated over the interval [0,2].
Ex:

$$y_1(t) = e^{-t} - e^{-2t} + e^{-3t},$$
$$y_2(t) = e^{-2t} - e^{-3t},$$
$$y_3(t) = e^{-3t} - e^{-t}$$

Discussion and Conclusion

In this study, we have derived the algebraic equations of order conditions for direct integrators of RKM for special fifth-order ordinary differential equations. Our approach is based on Taylor series expansion. We have derived two RKM methods of three-stage fifth-order, four-stage sixth-order respectively. Numerical results of the implementation show that the new integrators are as accurate as well-known existing methods; however, they are more efficient in implementation as they require less function evaluations. As such, we conclude that these

integrators are costly effective, in terms of computation time and number of evaluation than existing methods.

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Author's Contributions:

Mohammed S. Mechee: Analysis of the method, numerical simulation and writing.

Murtaza A. Kadhim: Analysis of the method, discussions and writing.

Ethics

There are no ethical issues associated with this work.

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