American Journal of Applied Sciences

Original Research Paper

q- $\mathfrak{sl}_2$ and Associated Wave and Heat Equations

Sami H. Altoum

Department of Mathematics, Academy of Engineering Science, Sudan
AL-Qunfudhah University College, Umm Al-Qura University, KSA

Article history
Received: 04-02-2018
Revised: 09-02-2018
Accepted: 08-05-2018

Email: samialtoum@hotmail.com

Abstract: For $q \in (0, 1)$, the q- deformation of the $\mathfrak{sl}_2$ Lie algebra (denoted by q-$\mathfrak{sl}_2$) is introduced and we give its representation. The heat and wave equations associated to the genrators of the q-$\mathfrak{sl}_2$ are studied.

Keywords: q-$\mathfrak{sl}_2$ Lie Algebra, Heat Equation, Wave Equation

Introduction

A Lie algebra $\mathfrak{g}$ (Erdmann and Wildon, 2006; Humphreys, 1978) is a vector space over a field $\mathbb{K}$ with an associated bilinear map $[.,.]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that the following hold:

- $[x; x] = 0$ for all $x \in \mathfrak{g}$
- $[x; [y; z]] + [y; [z; x]] + [z; [x; y]] = 0$ for all $x; y; z \in \mathfrak{g}$

The latter axiom of the above definition is called the Jacobi Identity. The idea of this axiom is to be a replacement for associativity, as we do not have that a Lie algebra is an associative algebra. We refer to this bilinear map $[.,.]$ as the Lie bracket of $\mathfrak{g}$. Let $\mathbb{K}$ be any field and let $\mathfrak{gl}(\mathfrak{n}; \mathbb{K})$ be the vector space of all $n \times n$ matrices defined over $\mathbb{K}$. Then $\mathfrak{gl}(\mathfrak{n}; \mathbb{K})$ is a Lie algebra with Lie bracket given by:

$$[x;y] = xy - yx \quad \forall x; y \in \mathfrak{gl}(\mathfrak{n};\mathbb{K}); \tag{1}$$

i.e., the commutator bracket. The special linear Lie algebra of order n (denoted $\mathfrak{sl}_n(\mathbb{K})$ or $\mathfrak{sl}(n, \mathbb{K})$ is the Lie algebra of $n \times n$ matrices with trace zero and with the Lie bracket given by (1). This algebra is well studied and understood and is often used as a model for the study of other Lie algebras. The Lie group that it generates is the special linear group. For $n = 2$, the space:

$$\mathfrak{sl}_2(\mathbb{K}) = \{s \in \mathfrak{gl}(\mathfrak{n};\mathbb{K}) | tr(s) = 0\} \subset \mathfrak{gl}(\mathfrak{n};\mathbb{K})$$

be the vector subspace of $\mathfrak{gl}(\mathfrak{n}; \mathbb{K})$ whose elements have trace 0 where $\mathbb{K}$ is any field. Now if $x; y \in \mathfrak{sl}_2(\mathbb{K})$ then we will have $[x; y] = xy - yx \times \mathfrak{sl}_2(\mathbb{K})$ hence the commutator brackets gives $\mathfrak{sl}_2(\mathbb{K})$ a Lie algebra structure, we denote $\mathfrak{sl}_2(\mathbb{K})$ by $\mathfrak{sl}_2$ for simplicity. As a vector space it can be shown that $\mathfrak{sl}_2(\mathbb{K})$ has a basis given by:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These elements have the following Lie bracket relations:

$$[e, f] = h,$$
$$[h, f] = -2f,$$
$$[h, e] = 2e.$$

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ plays an important role in the study of chaos and fractals, as it generates the Mobius group $S\,L(2, \mathbb{R})$, which describes the automorphisms of the hyperbolic plane, the simplest Riemann surface of negative curvature; by contrast, $S\,L(2, \mathbb{C})$ describes the automorphisms of the hyperbolic 3-dimensional ball. The simplest non-trivial Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$. Also, the $\mathfrak{sl}_2$ can be defined as the *-Lie algebra with three generators B^-, B^+, M and relations:

$$[B^-, B^+] = M, [M, B^\pm] = \pm 2B^\pm, (B^-)^*, \quad M^* = M.$$

Let $\mathfrak{g}_1$; $\mathfrak{g}_2$ be Lie algebras defined over a common field $\mathbb{K}$. Then a homomorphism of Lie algebras $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map of vector spaces such that $\varphi([x; y]) = [\varphi(x); \varphi(y)]$, i.e., it preserves the Lie bracket. A representation (Erdmann and Wildon, 2006; Humphreys, 1978) of a Lie algebra $\mathfrak{g}$ is a pair $(V; \varphi)$ where V is a vector space over $\mathbb{K}$ and $\varphi: \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism.

On the other hand, the language of q calculus (Abdi, 1962; Adams, 1929; Gasper and Rahman, 1990; Jackson, 1910; Leeuwen and Maassen, 1995) appeared. The natural number n has the following q-deformation:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, with [0]_q = 0.$$

Also, many important algebras was deformed using the q-calculus. Our question is, what is the q- analogue of the the $\mathfrak{sl}_2$?

As a response of this question, in this study, we introduce the q- deformation of the $\mathfrak{sl}_2$ Lie algebra (denoted by q-$\mathfrak{sl}_2$) is introduced. Moreover, as application, we study the heat and wave equations associated to the genrators of the q-$\mathfrak{sl}_2$.

The paper is organized as follows. In Section 2, we introduce the q-$\mathfrak{sl}_2$ and we give its representation. In Section 3, we study the heat equations associated to the generators of the q-$\mathfrak{sl}_2$. In Section 4, we study the wave equations associated to the generators of the q-$\mathfrak{sl}_2$.

q-$\mathfrak{sl}_2$

Definition 2.1

For $q \in (0, 1)$, the q-$\mathfrak{sl}_2$ is by definition the Lie algebra spanned by the operators A, B and C such that:

$$[A, C] = -2q^2 A$$
$$[B, C] = 2q^2 B$$
$$[A, B] = C$$

Theorem 2.1. (Representation of the $q - sl2$)

Let $q \in (0, 1)$, then we have:

$$[A_q, C_q] = -2q^2 A_q,$$
$$[B_q, C_q] = 2q^2 B_q,$$
$$[A_q, B_q] = C_q$$

where, A_q, B_q and C_q are given by:

$$A_q \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}, B_q \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, C_q \begin{pmatrix} q^2 & 0 \\ 0 & -q^2 \end{pmatrix}$$

Proof

We have:

$$A_q C_q \begin{pmatrix} 0 & -q^3 \\ 0 & 0 \end{pmatrix}$$

and:

$$C_q A_q \begin{pmatrix} 0 & q^3 \\ 0 & 0 \end{pmatrix}.$$

Then, we get:

$$[A_q, C_q] = \begin{pmatrix} 0 & -2q^3 \\ 0 & 0 \end{pmatrix}$$
$$= -2q^2 A_q.$$

On the other hand, we have:

$$B_q C_q = \begin{pmatrix} 0 & 0 \\ q^3 & 0 \end{pmatrix}$$

and:

$$C_q B_q = \begin{pmatrix} 0 & 0 \\ -q^3 & 0 \end{pmatrix}.$$

Then, we get:

$$[B_q, C_q] = \begin{pmatrix} 0 & 0 \\ 2q^3 & 0 \end{pmatrix}$$
$$= 2q^2 B_q.$$

Finally, we have:

$$A_q B_q = \begin{pmatrix} q^2 & 0 \\ 0 & 0 \end{pmatrix}$$

Then, we get:

$$[A_q, B_q] = \begin{pmatrix} q^2 & 0 \\ 0 & -q^2 \end{pmatrix}$$
$$= C_q$$

which completes the proof.

Heat Equations Associated to the Generator of q-$\mathfrak{sl}_2$

In this section, we will study the following three equations:

$$\begin{cases} \frac{\partial}{\partial t} ut = A_q ut \\ u_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, X_0, Y_0 \in \mathbb{C}. \end{cases} \qquad (2)$$

$$\begin{cases} \dfrac{\partial}{\partial t} ut = B_q ut \\ u_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, X_0, Y_0 \in \mathbb{C}. \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{\partial}{\partial t} ut = C_q ut \\ u_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, X_0, Y_0 \in \mathbb{C}. \end{cases} \tag{4}$$

Theorem 3.1

For $q \in (0, 1)$, the solution of the heat Equation (2) is given by:

$$u_t = \begin{pmatrix} qY_0 t + X_0 \\ Y_0 \end{pmatrix}$$

Proof

Let u_t given by:

$$u_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}.$$

Then, we get:

$$A_q u_t = \begin{pmatrix} qY_t \\ 0 \end{pmatrix}.$$

Therefore, we obtain:

$$\begin{cases} \dfrac{\partial}{\partial_t} X_t = qY_t \\ \dfrac{\partial}{\partial_t} Y_t = 0 \end{cases}$$

which implies that:

$$\begin{cases} \dfrac{\partial}{\partial_t} X_t = qY_t \\ Y_t = Y_0 \end{cases}$$

This gives:

$$\begin{cases} X_t = qY_0 + X_0 \\ Y_t = Y_0 \end{cases}$$

which completes the proof.

Theorem 3.2

For $q \in (0, 1)$, the solution of the heat Equation (3) is given by:

$$u_t = \begin{pmatrix} X0 \\ qX_0 t + Y_0 \end{pmatrix}$$

Proof

Let u_t given by

$$u_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}.$$

Then, we get;

$$B_q u_t = \begin{pmatrix} 0 \\ qX_t \end{pmatrix}.$$

Therefore, we obtain:

$$\begin{cases} \dfrac{\partial}{\partial t} X_t = 0 \\ \dfrac{\partial}{\partial t} Y_t = qX_t \end{cases}$$

This gives:

$$\begin{cases} X_t = X_0 \\ \dfrac{\partial}{\partial_t} Y_t = qX_t \end{cases}$$

which implies that:

$$\begin{cases} X_t = X_0 \\ Y_t = qX_0 t + Y_0 \end{cases}$$

Hence, we complete the proof.

Theorem 3.3

For $q \in (0, 1)$, the solution of (4) is given by:

$$u_t = \begin{pmatrix} X_0 e^{q^2 t} \\ Y_0 e^{-q^2 t} \end{pmatrix}$$

Proof

Let $q \in (0, 1)$ and u_t given by:

$$u_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}.$$

Then, we get:

$$C_q u_t = \begin{pmatrix} q^2 X_t \\ -q^2 Y_t \end{pmatrix}.$$

Therefore, we obtain:

$$\begin{cases} \frac{\partial}{\partial t} X_t = q^2 X_t \\ \frac{\partial}{\partial t} Y_t = -q^2 Y_t \end{cases}$$

which implies that:

$$\begin{cases} X_t = X_0 e^{q^2 t} \\ Y_t = Y_0 e^{-q^2 t} \end{cases}$$

This completes the proof.

Wave Equations Associated to the Generators of q-$\mathfrak{sl}_2$

In this section we are interested in the study of three wave equations associated to A_q, B_q and C_q.

Theorem 4.1

For $q \in (0, 1)$, the solutions of the following wave equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u_t = A_q u_t \\ u_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, X_0, Y_0 \in \mathbb{C} \end{cases}$$

are of the form:

$$u_t = \begin{pmatrix} \frac{1}{6}\alpha q t^3 + \frac{1}{2} Y_0 q t^2 + \beta t + X_0 \\ \alpha t + Y_0 \end{pmatrix}, \alpha, \beta \in \mathbb{C}.$$

Proof

Let u_t given by:

$$u_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}.$$

Then, since we have:

$$A_q u_t = \begin{pmatrix} q Y_t \\ 0 \end{pmatrix}$$

We get:

$$\begin{cases} \frac{\partial^2}{\partial t^2} X_t = q Y_t \\ \frac{\partial^2}{\partial t^2} Y_t = 0 \end{cases}$$

which implies that:

$$\begin{cases} \frac{\partial^2}{\partial t^2} X_t = q Y_t \\ Y_t = \alpha t + Y_0, \alpha \in \mathbb{C} \end{cases}$$

This gives:

$$\begin{cases} \frac{\partial}{\partial t} X_t = \frac{\alpha}{2} q t^2 + Y_0 q t + \beta \\ Y_t = \alpha t + Y_0, \quad \alpha\beta \in \mathbb{C} \end{cases}$$

Then we get:

$$\begin{cases} X_t = \frac{1}{6}\alpha q t^3 + \frac{1}{2} q Y_0 t^2 + \beta t + X_0 \\ Y_t = \alpha t + Y_0, \alpha\beta \in \mathbb{C} \end{cases}$$

This completes the proof.

Theorem 4.2

For $q \in (0, 1)$, the solutions of the following wave equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u_t = B_q u_t \\ u_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, X_0, Y_0 \in \mathbb{C} \end{cases}$$

are of the from:

$$u_t = \begin{pmatrix} \alpha t + X_0 \\ \frac{1}{6}\alpha q t^3 + \frac{1}{2} X_0 q t^2 + \beta t + Y_0 \end{pmatrix}, \alpha, \beta \in \mathbb{C}$$

Proof

Let u_t given by:

$$u_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}.$$

Then, since we have:

$$B_q u_t = \begin{pmatrix} 0 \\ qX_t \end{pmatrix}$$

We get:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X_t = 0 \\ \dfrac{\partial^2}{\partial t^2} Y_t = qX_t \end{cases}$$

which implies that:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} Y_t = qX_t \\ X_t = \alpha t + X_0, \alpha \in \mathbb{C} \end{cases}$$

This gives:

$$\begin{cases} \dfrac{\partial}{\partial t} Y_t = \dfrac{\alpha}{2} qt^2 + X_0 qt + \beta \\ X_t = \alpha t + X_0, \quad \alpha\beta \in \mathbb{C} \end{cases}$$

Then, we get:

$$\begin{cases} Y_t = \dfrac{1}{6}\alpha qt^3 + \dfrac{1}{2} qX_0 t^2 + \beta t + Y_0 \\ X_t = \alpha t + X_0, \alpha\beta \in \mathbb{C} \end{cases}$$

This completes the proof.

Theorem 4.3

For $q \in (0, 1)$, the solution of the following wave equation:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u_t = C_q u_t \\ u_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, X_0, Y_0 \in \mathbb{C} \end{cases}$$

is given by:

$$u_t = \begin{pmatrix} X_0 e^{qt} \\ Y_0 e^{iqt} \end{pmatrix}.$$

Proof

Let $q \in (0, 1)$ and u_t given by:

$$u_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$$

Then, since we have:

$$C_q u_t = \begin{pmatrix} q^2 X_t \\ -q^2 Y_t \end{pmatrix}$$

We get:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} q^2 X_t \\ \dfrac{\partial^2}{\partial t^2} Y_t = -q^2 Y_t \end{cases}$$

which gives:

$$\begin{cases} X_t = X_0 e^{qt} \\ Y_t = Y_0 e^{iqt} \end{cases}$$

This completes the proof.

Remark 1

In this study we introduced the q-$\mathfrak{sl}_2$. A q-deformation of some nuclear algebras of operators acting on space of holomorphic functions on a q-deformed complexification of real nuclear space can be studied and we expect to developp a new q-deforemed white noise theory to overcome the renormalisation problem, (Altoum *et al*., 2017; Ettaieb *et al*., 2012; 2014a; 2014b; 2016; Ouerdiane and Rguigui, 2012; Rguigui, 2015a; 2015b; 2016a; 2016b).

Acknowledgement

We would like to thank Dr. H. Rguigui for his help and guidance in this study. Also thank anonymous referees for their careful reading of our manuscript and helpful suggestions.

Ethics

The author declare that there is no conflict interests regarding the publication of this manuscript. This article is original and contains unpublished material.

References

Abdi, W.H., 1962. On certain q-difference equations and q-Laplace transforms. Proc. Nat. Inst. Sci. India Acad., 28: 1-15.

Adams, C.R., 1929. On the linear ordinary q-difference equation. Am. Math. Second Ser., 30: 195-205. DOI: 10.2307/1968274

Altoum, S.H., H.A. Othman and H. Rguigui, 2017. Quantum white noise Gaussian kernel operators. Chaos, Solitons Fractals, 104: 468-476. DOI: 10.1016/j.chaos.2017.08.039

Erdmann, K. and M.J. Wildon, 2006. Introduction to Lie Algebras. In: Springer Undergraduate Mathematics Series, Chaplain, M.A.J., A. MacIntyre, S. Scott, N. Snashall and E. Süli *et al*. (Eds.), Springer-Verlag.

Ettaieb, A., H. Ouerdiane and H. Rguigui, 2012. Cauchy problem and integral representation associated to the power of the QWN-Euler Operator. Commun. Stochastic Anal., 6: 615-627.

Ettaieb, A., H. Ouerdiane and H. Rguigui, 2014a. Powers of quantum white noise derivatives. Infinite Dimens. Anal. Quantum Probab. Related Top., 17: 1-16. DOI: 10.1142/S0219025714500180

Ettaieb, A., H. Ouerdiane and H. Rguigui, 2014b. Higher powers of quantum white noise derivatives. Commun. Stochastic Anal.

Ettaieb, A., N.T. Khalifa, H. Ouerdiane and H. Rguigui, 2016. Higher powers of analytical operators and associated $*$-Lie algebras. Infinite Dimens. Anal. Quantum Probab. Related Top., 19: 1-20. DOI: 10.1142/S0219025716500132

Gasper, G. and M. Rahman, 1990. Basic Hypergeometric Series. In: Encyclopedia of Mathematics and Its Application, Doran, R., M. Ismail, T.Y. Lam and E. Lutwak (Eds.), Cambridge University Press, Cambridge.

Humphreys, J.E., 1978. Introduction to Lie algebras and Representation Theory. Graduate Texts in Mathematics, Sheldon, A. and K. Ribet (Eds.), Springer-Verlag.

Jackson, H.F., 1910. q-difference equations. Am. J. Math., 32: 305-314. DOI: 10.2307/2370183

Leeuwen, H.V. and H. Maassen, 1995. A q-deformation of the Gauss distribution. J. Mathe. Phys. 36: 4743-4756.

Ouerdiane, H. and H. Rguigui, 2012. QWN-conservation operator and associated wick differential equation. Commun. Stochastic Anal., 6: 437-450.

Rguigui, H., 2015a. Quantum Ornstein–Uhlenbeck semigroups. Quantum Studies: Math. Foundat., 2: 159-175.

Rguigui, H., 2015b. Quantum λ-potentials associated to quantum Ornstein–Uhlenbeck semigroups, Chaos, Solitons Fractals, 73: 80-89. DOI: 10.1016/j.chaos.2015.01.001

Rguigui, H., 2016a. Characterization of the QWN-conservation operator and applications. Chaos, Solitons Fractals, 84: 41-48. DOI: /10.1016/j.chaos.2015.12.023

Rguigui, H., 2016b. Wick differential and Poisson equations associated to the QWN-Euler operator acting on generalized operators. Math. Slovaca.