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Nonstandard Analysis for Some Convergence Sequences Theories Via a Transfer Principle

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Abstract: The purpose of this article to introduce nonstandard models for a familiar types of convergent sequences theories through a transfer principle. The nonstandard analysis principle discuss, that any statements on areal system can be extended to a similar structure over larger hyperreal system therefore the results that hold true on the original system, remains true in a hyperreal system if and only if its $*$-transform is true. We apply such technique to transfer a classical proof for real sequences theorems, therefore we obtain an equivalents nonstandard proof on hyperreal system which is often clear, shorter and uncoblicated.

Keywords: Nonstandard Analysis, Hyperreal Numbers, $*$-Transform A Transfer Principle, Convergence Sequences

Introduction

Nonstandard analysis is anew mathematical technique that widely use in dever field of mathematics and other science such as Statistics and economics (Goldblatt, 1998). It become a powerful, mathematical tool in the 20th century for providing a new method in order to formulating statements and proving theorems which yelding for an enlarge view of the mathematical land scape (Goldbring, 2014; Goldblatt, 2012).

The first who has been succeeded to demonestrated a rigorous foundation for the use of infinitesimal and infinite numbers in analysis (Lengyel, 1996), and provided a basic concepts of nonstandard analysis was Abraham Robinson during 1960 s. Therefore a new large field of numbers known as a hyperreal system includes areal number system, infinite, infinitesimals numbers which are non-zero, infinitely larg and small numbers were constructed (Keisler, 2000). In fact a hyperreal numbers system $\mathbb{R}$ can be regarded as extension order field of areal numbers $\mathbb{R}^*$ areal number system infinitesimals. Moreovere A. Robinson has been showed that a relational structure over real numbers can be transferring to equivalent structure over hyperreal system (Hurd and Loeb, 1985). Therefore every statement holds true within real system remain true in hyperreal system hence this propriety for transferring statements known as a transfer principle which based on Robinsons approach (a nonstandard analysis).

There were a number of studies have examined a results on nonstandard analysis and its applications. Sun (2015) were introduced an applications economics, Similary (Duanmu, 2018) were applied the nonstandard analysis to Markov processes and Statistical Decision theory. An interesting application of a transfer principle for continuations of real functions to Levi-Civita field has been presented by (Bottazzi, 2018). A new approach to nonstandard analysis has been presented by (Abdeljalil, 2018), he proposed a very simple method in practice to nonstandard analysis without using the ultrafilter, (Ciurea, 2018) has constructed an approach for nonstandard analysis in a complete metric spaces.

Numerous studies have attempted to the framework for working with infinitesimals theory developed by Abraham Robinson and application for nonstandard analysis we reported them in (Mocanu *et al*., 2020; Sanders, 2019; Bell, 2019; de Jong, 2020; Ciurea, 2018; Goldbring and Walsh, 2019; Katz and Polev, 2017; Robinson, 2016).

We apply nonstandard analysis concepts to give nonstandard models equivalents for the classical theories of real sequences that converges. The method we used is a cording to Abraham Robinson for transferring properties from areal number system $\mathbb{R}$ to a hyperreal system $\mathbb{R}^*$ via a transfer principle therefore all properties of sequences of the real numbers were preserved. The equivalent proofs we obtained often simpler and directly which is desire result.

The study compose of five section we organized it as follows:

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The first section begin by giving a brief overview history of nonstandard analysis as introductory section. In section two show how a hyperreal system can be created, as an order field extension of real numbers and as ultra-power. A notations about a infinitesimal numbers and an ultra-power construction of the order field for the hyperreal system $\mathbb{R}^*$ were introduced.

In section three and four, we present an introduction to idea of theory for the language of relational structure in order to describe a transferring properties.

We formulate transfer principle and then introduce a first-order sentences which the transfer principle applies to by replacing statements by its $*$-transform. We the logical construction used in essentially way to transfer a statements from standard form into a nonstandard model.

The final section draws upon the entire study, we present a familiar concepts of convergence sequence theorem including, convergence theorem, limits theorem, cluster point, Cauchy sequence and a monotonic sequence theorem. We discuss their nonstandard model which equivalents to its standard formulas.

The Structure of the Hyperreal Numbers System

The hyperreal number can be constructed within two approach, the first one as an extension order field of a set of real numbers, and the other as ultra-power. The two approaches allows us to extend arbitrary functions and relations from $\mathbb{R}$ to $\mathbb{R}^*$.

The Structure of the Hyperreal Number System as Complete Order Field

A real numbers system is a complete order field $(\mathbb{R},+,.,0<)$ where +, · and < are the usual algebraic operation (relations of addition, multiplication and linear ordering) on $\mathbb{R}$. Recall that real system can be extend to an orderd field denoted by $\mathbb{R}^*$ which contains an isometric copy of $\mathbb{R}$ but strictly larger called a hyperreal or nonstandard system (Staunton, 2013; Hurd and Loeb, 1985). Furthermore the construction of hyperreal number is reminiscent of the construction of the reals from the rationales numbers by means of equivalence classes of Cauchy sequence (Hurd and Loeb, 1985) (Staunton, 2013). In fact an elements of hyeperreal number $\mathbb{R}^*$ should be viewed as infinite sequence of real numbers. For example, the sequence $\langle 1,2,3\ldots\ldots\rangle$ should represent some infinite element of $\mathbb{R}$. However, many different sequences of rational numbers represent the same real number $\mathbb{R}$.

Now we will introduce a basic arithmetical stricter of a hyper real and its relation to areal numbers.

Number Systems and Infinitesimal

An infinitesimal is known as a number that is smaller in magnitude than any non-zero real number and is larger than every negative real number or equivalently in absolute value it is smaller than $\frac{1}{m_0}$ for all $m_0 \in \mathbb{N}$ (Ponstein, 1975). Although zero is only infinitesimal number which belong to real number system, it may be extended in some way in order to include all infinitesimals (Ponstein, 1975). The following proposition illustrates some facts in a hyper areal numbers $\mathbb{R}$.

A non zero number k on $\mathbb{R}$ is said to be:

- Infinitely small or (infinitesimal) if, $|k| \leq \frac{1}{m_0}$, for every integer $m_0 \in \mathbb{N}$
- Finite if $|k| \leq m_0$ for some $m_0 \in \mathbb{N}$
- Hyper larger or (infinitely large) if $m_0 \in \mathbb{N}$, $\left|\frac{1}{k}\right| > m_0$, for all $m_0 \in \mathbb{N}$

A hyperreal numbers b is said to be positive infinitesimal if $b > 0$ but less than a number $o < a \in \mathbb{R}$, negative infinitesimal if $b < 0$ but its greater than a negative number $o < a \in \mathbb{R}$. A hyperreal numbers is said to be infinitesimal if it either positive or negative infinitesimal or zero (Goldblatt, 2012; Keisler, 1976). Hens the sets of all infinitesimals numbers to which zero belong and the set of all hyperlarg numbers which containing a classical numbers all together are constitute a nonstandard numbers therefore they are clearly an extension of the real numbers $\mathbb{R}$.

Hens the large system of numbers which contains areal numbers system, infinite, infinitesimals numbers that are non-zero, known as nonstandard or hyperreal numbers and denoted by $\mathbb{R}^*$.

Limited, Unlimited and Appreciable Numbers

Suppose $b \in \mathbb{R}^*$ and s, $t \in \mathbb{R}$ then the following propositions are holds,

Proposition. 2.3.1.

A number b is limited if $s < b < s$ for some numbers s, $t \in \mathbb{R}$.

Proposition. 2.3.2.

A number b is positive unlimited if $s < b$ for all $s \in \mathbb{R}$.

Proposition. 2.3.3.

A number b is negative unlimited if $b < s$ for all $s \in \mathbb{R}$.

Proposition. 2.3.4.

A number b is unlimited if it is positive and negative.

Proposition. 2.3.5.

A number b is positive infinitesimal if $0 < b < s$ for all $0 < s \in \mathbb{R}$.

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Proposition. 2.3.6

A number b is negative infinitesimal if $s < b < 0$ for all $0 > s \in \mathbb{R}$.

Proposition. 2.3.7

A number b is appreciable $s < |b| < t$ for some $s, t \in \mathbb{R}^+$. In facts all hyperreal numbers and an infinitesimals are (finite).

Proposition. 2.3.1

The product of an infinitesimal and a finite number is infinitesimal.

An integer m is limited exactly when it is standard and if m is nonstandard it said to be illimited, for example the rational $m = \frac{1}{x} \in \mathbb{Q}$ is limited without being standard, moreover its true that the rational $m = \frac{1}{x} \approx 0$ is infinitesimal when n is illumined. Here is some useful rules which dealing with preceding notations:

$$m \text{ is limited integer}, m_i \approx 0 (1 \le i \le n) \Rightarrow \sum_{1 \le i \le n} m_i \approx 0.$$

If $m = \frac{1}{x} \in \mathbb{Q}$ *and* $x \neq 0$ *then* $m \approx 0$, *i.e., m is infinite simal.*

If and only if $\frac{1}{x}$ is unlimited.

The terms finite and infinite are often used for limited and unlimited. Notice that b is limited for some $n \in \mathbb{N}$, unlimited iff for all, $|b|<n$, $n \in \mathbb{N}$, appreciable if only if $\frac{1}{n} < |b| < n$ for some $n \in \mathbb{N}$.

We now will introduce some basic a arithmetic operations on the hyper real numbers.

An Operations on the Hyperreal Numbers

Suppose a, b be infinitesimal, c, d be appreciable and e, f be unlimited numbers then the following operation are holds.

Addition

- $a + b \to$ Infinitesimal
- $a + c \to$ Appreciable
- $c + d \to$ Limited
- $e + a$, $e + c \to$ unlimited

Multiplication

- $a.b$, $a.c \to$ Infinitesimal
- $c.d \to$ Appreciable
- $c.e \to$ Unlimited
- $e.f \to$ Unlimited

Quotients

- $\frac{a}{c}, \frac{a}{e}$ *and* $\frac{c}{e} \to$ Infinitesimal
- $\frac{c}{d}$, $c \neq 0 \to$ Appreciable
- $\frac{c}{a}, \frac{e}{a}$, $a \neq 0 \to$ Unlimited
- $\frac{e}{c}$, $c \neq 0 \to$ Unlimited

Observe that the limited numbers and infinitesimals are each a subring of $\mathbb{R}^*$.

The Standard Part of Hyperreal Numbers

The standard part play a basic role in infinitesimal theorem, it connect between the finite numbers of nonstandard analysis and the classical numbers i.e., it rounds off each finite hyperreal to the nearest real number therefore infinite hyperreal numbers never possess standard part.

Definition. 2.5.1

Let $s \in \mathbb{R}$ be finite (or a limited number) then the unique real number y that is infinitesimally close to s, $(s \approx t)$ is called the standard part (or shadow) of x. we denote it by $st(s)$ or $sh(s)$. Thus we have $t \approx sh(s)$ or $(t \approx sh(sx))$.

Theorem. 2.5.1

Every limited hyperreal numbers s is infinitely closed to exactly one real number called the standard part (shadow) of s.

Closeness of Hyperreal Numbers

Definition. 2.6.1

A number x and y in $\mathbb{R}^*$ are said to be near or infinitesimal close, if their difference $x\text{-}y \in \mathbb{R}^+$ is infinitesimal, thus x is infinitesimal if and only if $x \approx y$. We said that x, y are finitely close if $x\text{-}y$ is finite and it written as $x \sim y$.

Every finite hyperreal number is infinitely close to exactly one real number, therefore existing of standard part of any infinite numbers depend on infinitely closeness to a finite.

The Monad and Galaxy Set of Hyperreal Number

For every hyperreal number, there exist two nonempty sets namely monad and galaxy which play an important role in infinitesimals theory.

In elementary Calculus, the pictorial device of an infinitesimal is used to illustrate part of a monad and an infinite telescope is used to illustrate part of an infinite galaxy (Stroyan and Luxemburg, 1977; Keisler, 2000; 1976).

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Definition. 2.7.1. (The Monad Set)

Let $x \in \mathbb{R}^+$ be a hyperreal number, the monad of *x* is the set, denoted by:

$$mon(x) = \{y \in \mathbb{R}^+ : x \approx y\}$$

Definition. 2.7.2. (The Galaxy Set)

The galaxy of *a* set *x* is the set:

$$gla(x) = \{y \in \mathbb{R}^+ : x - y \text{ is finite}\}.$$

Remark

I. The *monad*(0) is the set of infinitesimals and *gala*(0) is the set of finite hyperreal numbers.
II. Any are equal or disjoint.
III. Let be two monads *m*(*x*) and *m*(*y*) be two monads then they are ether:
   - Equal if $x \approx y$
   - Disjoint if $x \approx y$)

Proposition. 2.7.3

The relation $\approx$ is an equivalence relation on $\mathbb{R}^*$.

Proof

The relation $\approx$ equivalence relation if it satisfies the following condition.
For any finite hyperreal $x, y, z \in \mathbb{R}^+$, we have:

I. x-$x \approx 0$ is in $\mathbb{R}^*$.
II. x-$y \approx y$-x, so $x \approx y$ implies $y \approx x$
III. if x-y and y-z, in $\mathbb{R}^*$ then x-z is $\mathbb{R}^*$

Thus from I, II, III then ($\approx$) is an equivalence relation on $\mathbb{R}^*$.

Theorem 2.7.1

The set monad (0) of infinitesimal elements is a subring of $\mathbb{R}^*$ and an ideal in galaxy (0). That is:

(i). Sums, differences and products of infinitesimals are infinitesimal
(ii). The product of an infinitesimal and a finite element is infinitesimal
(For proof the see theorem (1.4) (Keisler, 1976))

Proposition 2.7.4

I. Two galaxies *G*(*x*) and *G*(*y*) are either equal if *x*-*y* is finite or disjoint
II. If $x \approx 0$ then *m*(*x*) is a translate of *m*(0)

Therefore for every $x \in \mathbb{R}^+$ then:

- $m(x) = \{y \in R: y = x + z, z \in m(0)\}$
- $G(x) = \{y \in R: y = x + z, z \in G(0)\}$

Corollary. 2.7.1

The quotient field *G*(0)/*m*(0) is isomorphic to the standard field of $\mathbb{R}^+$ (Andres and Rayo, 2015).

Proof

m(0) is the kernel of the linear (over $\mathbb{R}$) map st, i.e.:

$$m(x) = \{x \in G(0) : st(x) = 0\}.$$

The following theorem explain existence and uniqueness prosperity of standard part.

The Standard Part Principle and the Mapping

Theorem. 2.8.1. (Standard Part Principle)

Every finite number $x \in \mathbb{R}^+$ is infinitely close to a unique real number $y \in \mathbb{R}$. Therefore every finite monad contains a uniquely number on $\mathbb{R}$.

Proof

Let *x* be in $\mathbb{R}^+$ is infinitely close to a unique real number $y \in \mathbb{R}$. Then every finite hyperreal number *x* is infinitely close to a unique real number.

Uniqueness

Consider y, $z \in \mathbb{R}$ and $y \approx x$, $z \approx x$. Since $\approx$ is an equivalence relation we have $y \approx z$, hence y-$z \approx 0$. But *y*-*z* is in $\mathbb{R}$, so *y*-*z* = 0 and *y* = *z*.

Existences

Let $E = \{y \in \mathbb{R} : y < x\}$ be a nonempty set. A set *E* has an upper bound if there is real number $y > 0$ such that: $|x| < y$, whence $-y < x < y$ so, $-y \in E$ hence *y* is an upper bound of *E*. Since $\mathbb{R}$ is complete ordered field, so the set *E* has at least upper bound *h*. For every $y \in \mathbb{R}$ where $y > 0$ we have:

$$y \le x - h \le y,$$

It follows that x-$h \simeq 0$. Hence $x \simeq h$ (Keisler, 1976).

Definition. 2.8.2. (The Standard Part Map.)

The map *st*: $x \rightarrow st(x)$ called the standard part map. Clearly maps *x* onto *st*(*x*) since $st(x) = x$, when $x \in \mathbb{R}$, furthermore it preserves algebraic structure as in the following theorem (Andres and Rayo, 2015).

Theorem 2.8.3

For every $x, y \in \mathbb{R}^+$:

i. $st(x \pm y) = st(x) \pm st(y)$
ii. $st(x.y) = st(x).st(y)$
iii. $x \leq y$ *implies* $st(x) \leq st(y)$
iv. $st(|x|) = |st(x)|$
v. $x < y$ implies $st(x) = st(y)$, iff $x\text{-}y \in \mathbb{R}^+$ *or* $(x \simeq y)$
vi. $st(\max(x, y)) = \max(st(x), st(y))$
vii. If $a \in \mathbb{R}$, then $st(a^*) = a$

The existence of standard of limited numbers follows from the Dedekind completeness of areal numbers $\mathbb{R}$. In fact the existence of standard part is a tentative formulation of completeness.

Theorem. 2.8.4

Every limited hyperreal $x \in \mathbb{R}^+$ is infinitely close to exactly one real number implies the completeness of $\mathbb{R}$ (Goldblatt, 2012) [theorem (5.8.1)].

In the following section is we will show the constructing of a hyperreal number system as linearly ordered field based on the ultra powers construction of nonstandard model.

A Construction of Hyper Real Number System as Ultra-Power

The construction of *hyperreal number* $\mathbb{R}^+$ from the a real *number* $\mathbb{R}$ is similar to construction of a real from the rational numbers $\mathbb{Q}$ by means of equivalence class of Cauchy sequences (Staunton, 2013). To construct a hyperreals $\mathbb{R}^*$, first we illustrate some notion of an ultrafilter, which will allow us to do a typical ultra power construction of the hyperreal numbers.

Suppose $\mathbb{R}^{\mathbb{N}}$ for $\mathbb{N} = \{1,2,\ldots..\}$ be the set of real-valued sequences, under point wise addition and multiplication. Let $u = (u_i)$, $v = (v_i)$ are elements in $\mathbb{R}^{\mathbb{N}}$ which defined as follows:

i. $u \oplus v = u_i \oplus v_i,\ i \in \mathbb{N}$
ii. $u \odot v = u_i \odot v_i,\ i \in \mathbb{N}$

However, $u \oplus v$ and $u \odot v$ are in $\mathbb{R}^{\mathbb{N}}$, so *its* is closed under point wise addition and multiplication. So ($\mathbb{R}^{\mathbb{N}}$, $\oplus$, $\odot$) is a commutative ring with identity sequence, however, $\mathbb{R}^{\mathbb{N}}$ satisfies all the properties of a field with identity, (1,11,1,…) and (0,0,0,…) = 0 and additive inverse consider, for example, the two sequences u = (0,1,0,1,…), v = (1,0,1,0…) neither of *u or v* equal to the zero. However, point wise multiplication would give us:

$$u.v = (0,1,0,1,\ldots) \odot (1,0,1,0\ldots) = 0.$$

Thus two nonzero elements u, v whose product is zero, are prevent the sequence $\mathbb{R}^{\mathbb{N}}$ to be an order field. To avoid this problem and to introduce equivalent relation which make $\mathbb{R}$ into an order field and then ex tend it to the hyperreals $\mathbb{R}^*$, we must construct a hyperreal number system $\mathbb{R}^*$ as an ultrapower of the real number system (Hurd and Loeb, 1985). To present an equivalence relation we need the notion of an ultrafilter to do so, first we must present a definition of a filter (Staunton, 2013).

Definition. 2.3.1. A Filter

Let Γ be a nonempty set of $\mathbb{N}$. A filter on Γ is a nonempty collection Ω of Γ having the following properties (Staunton, 2013):

i. The empty set $\varnothing \notin \Gamma$
ii. If U, $V \in \Omega$ then $U \cap U \in \Omega$
iii. If $U \in \Omega$ $\mathbb{N}$ and $\supseteq V \supseteq U$, then is cofinite or $V \in \Omega$

Definition. 2.3.2. A Free filter

If all elements of a filter are infinite sets then it said to be free (or non-principal).

Proposition. 2.3.1.

a. Every Ω filter contains the nonempty set Γ
b. $\sigma = \{\Gamma\}$ is smallest filete on Γ
c. A filter Ω is apropper if $\varnothing \notin \Omega$
d. A filters are closed under finite intersection

Definition 2.3.3. An Ultra Filter

A filter Ω is said to be an ultrafilter denoted by μ iff any subset U of Γ either U or $U^c \in \Omega$ (not both by(i), (ii)) where:

$$U^c = (\Gamma - U) \in \Omega$$

If Γ is an infinite set the collection $y_l = \{A \subseteq I\text{: } I\text{-}A$ is finite$\}$ is a filter called the cojnite or Frichet filter on Γ.

Proposition. 2.3.2

Let U, $V \in \mu$ and $U^c \in \mu$ be a complement of U then the following properties are holds:

i. $U \cap V$, $U \cup V \in \mu$ iff U, $V \in \mu$
ii. $U^c \in \mu$ iff $U \notin \mu$
iii. An ultraflter μ is an ultrafilter on Γ iff Γ is amaximal proper filter

Definition. 2.3.4. The Fre'chet Filter

If Γ is an infinite set then collection:

$$\xi = \{U \subseteq \Gamma : \Gamma - U \text{ is finite}\}$$

is called a Fre'chet filter or cognate.

The Fre'chet filter ξ is not an ultrafilter Moreover its proper if Γ is infinite. An ultrafilter μ on Γ is free if it contains ξ. Hence a nonprincapl ultrafilter must contain all a finite sets. This is a critical property used in construction infinitesimals and infinitely large numbers. Here are some important properties of μ (Staunton, (2013).

Definition.2.3.5. Free Ultra Filter

Combining definitions (2.3.3) and (2.3.4) we come up with the definition of a free ultrafilter.

Also μ is said to be free if it contains the Fre'chet filter.

A free ultrafilter δ on Γ contain any finite set of Ω. Moreover, all elements of the δ are infinite sets (Davis, 2009).

The free ultrafilters doesn't always exist. Hence, an ultrafilters are important for the purpose of construction of hyperreal $\mathbb{R}^*$.

An infinite sequences of real numbers are represent a free ultrafilter often use to give a rules for equality and identification, so we can come up with a mathematically consistent and sensible system of hyperreal numbers therefore in this way a hyperreal number can be generated.

An Equivalence Relation on Real Valued Sequence

The relation ≡ on $\mathbb{R}^{\mathbb{N}}$ is called an equivalence relation if it is -is reflexive ($u \equiv u$), symmetric ($u \equiv v \Rightarrow v \equiv u$) because ≡ is a symmetric relation on $\mathbb{R}$ and transitive ($u = v$ and $v = w$ imply $u = w$) because of conditions (ii) and (iii) for a filter. The equivalence relations are use the notation $u \sim v$.

The set $\mathbb{R}^{\mathbb{N}}$ can be divided into disjoint subsets (called equivalence classes) by the relation ≡. Each equivalence class consists of all sequences equivalent to any given sequence in the class, therefore u and v are said to be in the same equivalence class iff $u = v$. Two sequences which differ at only a finite number of places are equivalent under ≡ (Hurd and Loeb, 1985).

The Equivalence Classes on a Real Valued Sequence and an Ultrapower

In order to extend the real numbers system $\mathbb{R}$ to the hyperreal $\mathbb{R}^*$ in an ultrapower concept we can use infinite sequences of real numbers (Davis, 2009) before doing so, we shall create a field of real-valued sequences, in which every standard real numbers are embedded as the corresponding constant sequence.

Let M denote the set of all the quivalence classes of $\mathbb{R}^{\mathbb{N}}$ in deuced by ≡. The equivalence class containing a particular sequence $u = (u_i)$ is denoted by $[u]$ or u. Thus if $u = v$ in d then $u = [u] = [v] = v$.

We can define a relation ≡ on $\mathbb{R}^{\mathbb{N}}$ by putting:

$$\langle u_i \rangle = \langle v_i \rangle \text{ if and only if}$$
$$\{i \in \mathbb{N} : u_i = v_i\},$$

When this relation holds it may be said that the sequences $u_i = v_i$ possess same values at almost i.

Elements of M are called nonstandard or hyperreal number s and technically its known as an ultrapower (Goldblatt, 2012).

Lemma. 2.5.1

The relation ≡ is an equivalence relation on a hyperreal $\mathbb{R}^{\mathbb{N}}$.

Let M denote the set of all equivalence classes of $\mathbb{R}^*$ in duced by ≡. The equivalence class containing a particular sequence $u = (u_i)$ is denoted by $[u]$ or u. Thus if $u \equiv v$ in $\mathbb{R}^*$ then $u = [v] = [u] = u$. Elements of $\mathbb{R}$ are called nonstandard or hyperreal numbers. We use the same idea to define operation $u \equiv v$ and relations which make $\mathbb{R}$ into an ordered field.

Equivalence class of a sequences. u, $v \in \mathbb{R}^{\mathbb{N}}$ under relation ≡ denoted by $[u]$, $[v]$ respectively thus:

$$[u] = \{v \in \mathbb{R}^{\mathbb{N}} : u \equiv v\}$$
$$[v] = \{u \in \mathbb{R}^{\mathbb{N}} : v \equiv v\}$$

We now define an operations and relations which we will used it to make $\mathbb{R}$ into an ordered field.

Definition. 2.5.1

Let $u = [\langle u_i \rangle]$ and $v = [\langle v_i \rangle]$ then:

i. $u + v = [\langle u_i + v_i \rangle]$, i.e., $[u] + [v] = [u \oplus v]$
ii. $u.v = [\langle u_i.v_i \rangle]$, i.e., $[u].[v] = [u \odot v]$
iii. $[u] < [v]$ iff $[u < v] \in \mu$ iff $\{i \in \mathbb{N} : u_i < v_i\}$

So the equivalence class of the sequences of n-th term is given by:

$$[u_i] + [v_i] = [u_i \oplus v_i] = [u_i + v_i],$$
$$[u_i].[v_i] = [u_i \odot v_i] = [u_i.v_i].$$

Not that the quotient set of $\mathbb{R}^{\mathbb{N}}$ under ≡ can denoted by $\mathbb{R}^* = \{[u] : u \in \mathbb{R}^{\mathbb{N}}\}$.

Remarks

$\mathbb{R}$ is technically known as an ultra-power. We have used the same idea to define operations and relations which make $\mathbb{R}$ into an ordered field (Goldblatt, 2012).

Logically for u, $v \in \mathbb{R}^{\mathbb{N}}$ we ca n replace the set, $\{r \in \mathbb{N} : u_i = v_i\}$ by $[\![u = v]\!]$ thus $u \equiv v$ iff $[\![u = v]\!] \in \mu$.

Now since we have all the necessary tools so that we are ready to show that that $\mathbb{R}^*$ is order field.

Theorem.2.5.1

The structure $\mathbb{R}^*$ is a linearly ordered field.

Proof

That $\mathbb{R}^*$ is a commutative ring with zero 0 = [⟨0,0,0,….⟩] and unit 1 = [⟨1,1,1,…..⟩] and additive invers given by -[⟨u_i: $i\in\mathbb{N}$ ⟩] = [⟨-u_i: $i\in\mathbb{N}$ ⟩] (i.e., if $u \neq 0$ then there is an element A, B, $\mu^{-1}\in\mathbb{R}^{\mathbb{N}}$ so that $u.u^{-1} = 1$). To show that it has multiplicative inverses, suppose $[u] \neq [0]$. Then $\{i\in\mathbb{N} : u_i = 0\} \notin \mu$ and so $\{i\in\mathbb{N} : u_i = 0\}\in\mu$, since μ is ultrafilter, hence $w = \{i\in N: u_i \neq 0\}\in\mu$ is ultrafilter. Define $u^{-1} = [\langle u_i\rangle]$ where $[u_i] = u_i^{-1}$ if $u_i \neq 0$ and $[u_i] = 0$ i.e.:

$$u_i = \begin{cases} \frac{1}{u_i}, & if\ i\in w \\ 0, & otherwise \end{cases}$$

Then $[\![u\odot v = 1]\!]$ is equal to w so:

$$u\odot v = 1 \in \mu.$$

Let $r\odot s \equiv 1$ then $[u]\cdot[v] = [u\odot v] = [1]$ is in $\mathbb{R}^*$ hence $[v]$ is multiplicative inverse of $[u]^{-1}$ or $[u]$. Now let $A = [\![u < v]\!]$, $B = [\![u = v]\!]$, $C = [\![v < u]\!]$ we want to show that $\mathbb{R}^*$ is a linearly ordered field with the ordering given by < . · Hence exactly one of:

$$[u]<[v],\quad [u]=[v],\quad [u]<[v]$$

Is true. The element u of $\mathbb{R}$ is positive if $u > 0$. We must show that the set $\{[u]: [0] < [u]\}$ of positive elements in a hyperreal $\mathbb{R}^*$ is closed under addition, multiplication and the law of Trichotomy which states for a given element u either:

$$u>0\ or\ u=0,\ or -u>0$$

(where, $-u$ is the additive inverse of u) (Law of trichotomy). To demonstrate suppose $[u] = [\langle u_i\rangle]$ and define:

$$L=\{i\in\mathbb{N}:u_i>0\}, M=\{i\in\mathbb{N}:u_i=0\},$$
$$N=\{i\in\mathbb{N}:u_i=0\}$$

We want to show that only one of k, m and n is in μ, from the law of trichotomy in $\mathbb{R}$, we see that $k\cup m\cup n = \mathbb{N}\in\mu$. Now one of l, m and n are in μ also k^c, m^c, n^c are in μ also:

$$(k\cup m\cup n)^c = (k^c\cap m^c\cap n^c) = \varnothing\in\mu$$

which is a contradiction.

Clearly the fact that $\mathbb{R}^*$ is totally ordered follows from the fact that $\mathbb{R}$ is totally ordered and that μ is an ultrafilter. Thus, we have that $\mathbb{R}^*$ is a totally ordered field. Hence a hyperreal numbers is totally orderd field $\mathbb{R}^*$ numbers (Hurd and Loeb, 1985).

The filed $\mathbb{R}$ is embaded into $\mathbb{R}^*$ through a mapping that assign to each $u\in\mathbb{R}$, the hyperreal $u^*\in\mathbb{R}^*$ denoted by the equivalence class of the constant sequence with value u we shall identify u and u^*.

Now we want show that areal number system $\mathbb{R}$ can be embedded isomorphic as linearly ordered sub field of $\mathbb{R}^*$.

One can relate areal number $u\in\mathbb{R}$ with constant sequence $u = [\langle u, u, u,\ldots\ldots\ldots.\rangle]$ assign to the $\mathbb{R}^*$ elements. Define $*(u) = u^*$ where, $u^* = [u] = [\langle u, u, u,\ldots\ldots\ldots.\rangle]$. For u, $v\in\mathbb{R}$ we have the following properties:

$$(u+v)^* = u^*+v^*$$
$$(u\cdot v)^* = u^*\cdot v^*$$
$$u^*<v^*\ iff\ u<v$$
$$u^*=v^*\ iff\ u=v$$

Theorem. 2.5.2

The map $*$: $u \to u^*$ is an order preserving field isomorphism from $\mathbb{R}$ to $\mathbb{R}^*$.

Proof

The mapping $*$ is one to one for $u^* = v^*$ then, $[\langle u, u, u,\ldots\ldots\ldots.\rangle] = [\langle v, v, v,\ldots\ldots\ldots.\rangle]$ and so $u = v$. To explain that a mapping *Preserves the field order properties we must show that:

i. $[\langle u, u, u,\ldots\ldots\ldots.\rangle] + [\langle v, v, v,\ldots\ldots\ldots.\rangle] = [\langle u + v, u + v, u + v,\ldots\ldots\rangle]$
ii. $[\langle u, u, u,\ldots\ldots\ldots.\rangle].[\langle v, v, v,\ldots\ldots\ldots.\rangle] = [\langle u.v, u.v, u.v,\ldots\ldots\ldots.\rangle]$, establishes, $(u + v)^* = u^* + v^*$ for (i) and $(u.v)^* = u^*.v^*$ for (ii) respectively

The details are left to the readers.

This results is useful for identify the real number u with u^* to regard $\mathbb{R}$ as Subfield of $\mathbb{R}^*$. In particular we may identify [0] with 0 and [1] with 1 (Hurd and Loeb, 1985; Goldblatt, 2012).

Proposition. 2.5.3.

Given $*$: $\mathbb{R}\to\mathbb{R}^*$, we define a map ρ on entities by $G^\rho = \{g^\rho: g\in G\}$, the map ρ gives the embedded standard copy of G. The $*$ is superstructure homomorphism of $\mathbb{R}$ if and only if it is a one to one map on $\mathbb{R}^*$. Hence the $*$ preserves the following,

- $\in$ (read elemnt of): If G is entity of $\mathbb{R}^*$ and $\in G$, then $g^*\in G^*$
- = (equality): If G is an entity then $\{(u, u): u\in G\}^* = \{(v, v): v\in G^*\}$

- A finite set: $\{g_1, g_2, \ldots g_n\}^* = \{g_1^*, g_2^* \ldots\ldots g_n^*\}$ for $g_1, g_2, \ldots g_n \in \mathbb{R}$
- Set operations: $\varnothing = \varnothing^*$, $(G \cup K)^* = G^* \cup K^*$, $(G \cap K)^* = G^* \cap K^*$, $(G/K)^* = G^* \backslash K^*$, $(G \times K)^* = G^* \times K^*$. G, K are entities
- The domain and ranger of n-aray relations and commutes with permutations of the variables. $(dom(\varnothing))^* = dom(\varnothing^*)$, $(rang(\varnothing))^* = rang(\varnothing^*)$
- Atomic standard definitions of set $\{(u, v): u, v \in G\}^* = \{(c, w): v, w \in G^*\}$.

Moreover $*$ produces a proper extension $G^* \supset G^\rho$ with equality iff G is a finite set. Those properties imply that the image $\mathbb{R}^*$ is a nonstandard model in the formal logic sence which we will describe below.

Remark

Therefore we conclude the fact that:

- The $\mathbb{N}^*, \mathbb{Z}^*, \mathbb{Q}^*$ and $\mathbb{R}^*$ are extensions of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}^*$, respectively
- The hyperreal extension $\mathbb{R} \to \mathbb{R}^*$ preserves all order properties of an ordered fields, hence a real numbers form of a hyperreal numbers and the order relation. Therefore $\mathbb{R}^*$ is an ordered field extension of $\mathbb{R}$

The following principle is a necessary for extending sequences and functions to the hyperreals (Davis, 2009).

The Extending Principle

Each function in the standard models can be extended it to a function acting on the corresponding nonstandard models. To be accurately for every real function f of one or more variables there is a corresponding hyperreal f^* of the same numbers of variables, f^* denoted the natural extension of f.

We can extend real-valued functions to hyperreal-valued functions in the following ways, let $f: \mathbb{R} \to \mathbb{R}$ be a function then for every real-valued sequence $\mathbb{R}^{\mathbb{N}}$, let $\langle f(u_1), f(u_2), \ldots\ldots\ldots \rangle$ for $u_i \in \mathbb{R}^{\mathbb{N}}$, $i \in \mathbb{N}$ then the extended function $f^*: \mathbb{R}^* \to \mathbb{R}^*$ by can defined as:

$$f^*([\langle u \rangle]) = [\langle f(u) \rangle]$$

for any hyperreal number $[u] \in \mathbb{R}^*$. In other words:

$$f^*([\langle u_1, u_2, ,,, \rangle]) = [\langle f(u_1), f(u_2), \ldots \rangle]$$

If there is a function f: $L \to \mathbb{R}$ where $L \in \mathbb{R}$, to extend the function f to the hyperreals, we have to define the extension of its domain L to a subset L of the hyperreal (Davis, 2009). We define the extension L^* of a subset L of areal to be the set:

$$L^* = \{[u] \in \mathbb{R}^* \mid u_i \in L, \forall i \in \mathbb{N}\}$$

Therefore L is the set of equivalence classes of sequences whose values range over the elements of L. Then the a function $f: \mathbb{R} \to \mathbb{R}$ can extend to a function $f^*: \mathbb{R}^* \to \mathbb{R}^*$ therefore such extension processes same rules as the original functions and relations.

The Formal Language of Relational Structure

In this section we will give an elementary idea on the language theory of relational structure with a few examples.

We will use a formal logical symbol to express statements that were asserted to be true or false of the structure $\mathbb{R}$ and $\mathbb{R}^*$.

The Simple Formal Languages for Relational Systems and the Logical Structure of First Order Statement

Definition. 3.1.1. (Relational Structure)

A relational structure is a system $\theta = \{\theta, q, f\}$ consists of a nonempty set θ, a collection of finitary relations q on θ and a set f functions relations on θ.

Definition. 3.1.2

A language L is a set that including all logical symbols and quantifiers (including the equality sign and the parenthesis) and some arbitrary number of constants, variables, function symbols and relation symbols (Stroyan and Luxemburg, 1977).

In order to describe a formal language it is first necessary to describe the symbols of the language and then we can describe the process of forming sentences. each relational stricter θ then L is language L_θ is bases in the logical symbol. Therefore any statement that is expressible in logic structure is mentioning only standard numbers is true in $\mathbb{R}$ if and only if it is true in $\mathbb{R}^*$.

The basic symbols is divided into two types:

(1). The symbols consists of logical symbols which are common to any simple language and do not vary if the statements is changed as containing the following symbols:

- Logical connective: $\wedge$(and), $\vee$(or), $\rightarrow$(inplies) $\leftrightarrow$(if f and only if), $\neg$(not). With their usual interpretation
- Quantifier symbols: $\forall$(for all), $\exists$(there exists)
- Operation, relation and function symbols, P, f, $*$, et cetera
- Parentheses and bracketes (,) and [,], $\langle , \rangle$

- An infinite list of constants which formed asset of Symbols which usually denoted by Roman or Greek letters
- The basic predicates are $\in$ and =
- An infinite list of variable: Which is countable collection of symbols in which we use letters *x*, *y*, z,……
- An infinite list of function symbols: *f*, *g*, *h*, ………
- An infinite list of relation symbols: *P*, *R*, *Q*, ……….

(2). The symbols in the second category depend on *Q* and will be called parameters. They consist of constant Symbols, Relation Symbols, function Symbols

The Formula and the θ-Term

There are expressions like composite functions in usual mathematical notation, constant, variable and function symbols is a string of symbols from the alphabet, they are special cases of terms we denoted L_θ-term and they are defined inductively by the following laws:

- Each constant symbol is an L_θ- term.

  i. Each variable symbol is an L_θ- term.
  ii. If *f* is the name of a function of *n* variables and $\tau_1,\ldots,\tau_n$ are L_θ-terms, then $f(\tau_1,\ldots,\tau_n)$ is an L_θ-terms

For example $f(2, g(x, y))$ and $\cos(x + y)$ are L_θ- terms

Definition. 3.2.1. (A Closed Term)

A closed term is term which that made up of constant and functions symbols. It is undefined if it does not name anything.

Remark

- A term containing no variables is called a constant term
- A closed term is on that has no variables and therefore made up of constants and functions symbols. A closed term is undefined if it does not name any things. For example the constant τ names itself and $f(\tau_1,\ldots,\tau_n)$ is undefined if one of $\tau_1,\ldots,\tau_n$ is undefined

Definition.3.2.2. (A formula)

If ϕ, ψ are L_θ-formula then it follows that:

- If ϕ is a formula so is $\neg\phi$. If ϕ and ψ are formulas, then so is $(\phi\wedge\psi)$
- If ϕ is a formula and *x* is a variable then $(\exists x)\phi$ is also a formula
- If ϕ is L_θ-formula and *x* is any variables symbol and P is subset of *Q* then, $(\forall x\in P)\phi$, $(\exists x\in P)\phi$ are L_θ -formula

Now we will give some basic concepts in mathematical logic.

A Sentences and an Atomic Relation

Definition. 3.3.1. (A Sentences)

A sentence known as a formula in which all variables are bounded. If the closed terms of the sentence are all defined then it has a fixed meaning and it is either true or false.

Definition. 3.3.2. (A Free Variables)

A free variable is a variable which obtained by Replacing any variable occurring in a statement by some constant to obtain another meaningful statement.

Definition. 3.3.4. (Bounded Variables)

A variable that is not free is called a bounded or dummy variable.

Definition. 3.3.5. (A Tomic Sentences)

An atomic sentences is a formula which has no variables and written are of the form $Q(\tau_1,\ldots,\tau_n)$, where *Q* is *n*-ary relation and $(\tau_1,\ldots,\tau_n)$ resents terms' of the sentences. A sentences also may defined inductively (Goldblatt, 2012).

Definition 3.3.6. (A Simple Sentences)

A simple sentences known as a language takes two types denoted by, atomic and a compound sentence and consists of a basic and combinations symbols which defined as a string of symbols in a sentence.

An Atomic Relation

An atomic relations are simplest mathematical relations by which are meant relations containing neither logical connectives nor quantifiers, hence relations such, as (=, $\in$,…etc.) (Staunton, 2013).

Atomic relations can be regarded as functions be {*true*, *false*}. A sentences are of the form $Q\langle\tau_1,\ldots,\tau_n\rangle$ where *Q* the name of an *n*-ary relation and the τ^i(I = 1,…,*n*) are constant terms therefore $\tau_1,\ldots,\tau_n$ are all closed i.e., there is no variables in the formula (Goldblatt, 1998).

An arbitrary statement which composed of a finite number of atomic relations and logical connectives can be define it inductively as follows, if ϕ and ψ are two sentence, then the following are also sentences:

- $\neg\phi$
- $\phi\wedge\psi$
- $\phi\vee\psi$

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- $\phi \to \psi$
- $\phi \leftrightarrow \psi$
- $(\forall x)\phi$
- $(\exists x)\phi$

Not that all the logical connectors can be derived from logical symbols ¬, ∧, ∀ as follows:

- $(\psi \vee \phi) = \neg((\neg\psi)\wedge(\neg\phi))$
- $(\psi \to \phi) = \neg(\psi \wedge (\neg\phi))$
- $(\psi \leftrightarrow \phi) = \neg(\psi \wedge (\neg\phi)) \wedge \neg(\phi\wedge(\neg\psi))$
- $(\forall x)\phi = \neg(\exists x)\phi$

Example. 3.4.1

The first order field axioms can be expressed as first order logic statements as follows:

- **Associativity Prosperity**:

$$(\forall x)(\forall y)(\forall z)(x+(y+z)=(x+y)+z)$$
$$(\forall x)(\forall y(\forall z)(x\cdot(yz)=(xy)\cdot(z))$$

- **Commutative Prosperity:**

$$(\forall x)(\forall y)(x+y=y+x)$$
$$(\forall x)(\forall y)(x-y=y\cdot x)$$

- **Distributive Prosperity:**

$$(\forall x)(\forall y)(\forall z)(x\cdot(y+z)=x\cdot y+x\cdot z)$$

- **Existence of Identities:**

$$(\exists x)((0=x)\wedge(Ay)(x+y=y))$$
$$(\exists x)((1=x)\wedge((Ay)(x\cdot y=y))$$

- **Existence of Inverses:**

$$(\forall x)(\exists x)(x+y=0))$$
$$(\forall x)(x\neq 0\to\exists y)(x\cdot y=1)$$

- **Total ordering Property:**

$$(\forall x)(\forall y)((x\le y)\wedge(y\le x))\to(x=y))$$
$$(\forall x)(\forall y)(\forall z)(((x\le y)\wedge(y\le z)\to(x\le z))$$
$$(yx)(\forall y)(((x\le y)\bar{V})(y\le x))$$

We now will introduce the notion of the ∗-transform of first-order L_θ sentence which is useful tool for transforming L_θ-sentences in areal $\mathbb{R}$ to the L_{θ^*} sentence in a hyperreal $\mathbb{R}^*$.

The Truth Value of Sentences

Recall that a sentence is either true or false in the real number system. Let are two sentence with standard meaning of symbolic connectives ∧, ∨, →, ↔, ¬ we will present some rules that usually using for calculation a truth values of a sentences:

- $\phi\wedge\psi$ are true if ϕ are true and ψ are true
- $\phi\vee\psi$ are true if ϕ are true ore ψ are true
- $\neg\phi$ are true if ϕ are false
- $\phi \to \psi$ is true if and only if either ϕ is false or else ψ is true
- $\phi \leftrightarrow \psi$ is true if and only if ϕ and ψ either both true ore both false

The mathematical formulation for statements associated with truth values rules able use to distinguish exactly which property is can transferable from $\mathbb{R}$ to $\mathbb{R}^*$ and vice- versa.

The Transfer Principle

The transfer principle is the powerful tool that allows us to use the methods of non-standard analysis to prove results in standard analysis (Staunton, 2013).

The transfer principle states that” a formula is true on areal system $\mathbb{R}$ if and only if the corresponding formula is true on $\mathbb{R}^*$ ".hence the transfer principle allow us to show that a hyperreal $\mathbb{R}^*$ has all the properties of $\mathbb{R}$ and also we can prove theorems about $\mathbb{R}$ by first proving them in $\mathbb{R}^*$ on the other words a transfer principle extends all a classical rules on a reals system to the hyperreal system which allow for easier and more intuitively natural proofs in a hyperreal system (Davis, 2009)**.**

The ∗-Transforms for First-Order Sentences

Definition 4.1.1

The *-transform of a simple sentence Q in L_θ-formula is the simple sentence Q^* in L_{θ^*} -formula obtained by starring all function and relation symbols in the sentence Q.

Thus, constructing a ∗-transform of a sentence L_θ really just consists of putting a ∗ on every term in Q, putting a ∗ on any relation symbol in Q and putting a ∗ on every set in Q acting as a bound on a variable (Davis, 2009; Keisler, 1976; Goldblatt, 2012).

Notation

Note that the ∗-transform arises by attaching the prefix to symbols but not attaching to variable symbols (Goldbring, 2014).

First we introduce a number of example which illustrate how we the a statements can be transforming.

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Example. 4.1.1

Any positive real has areal n-th root for all $n\in\mathbb{N}$.

This statements can be formulated as:

$$(\forall n\in\mathbb{N})(\forall x\in\mathbb{R}^{+})(\exists y\in\mathbb{R})(\sqrt[n]{x}=y). \tag{1}$$

Which is true. We can transform it to the true sentence:

$$(\forall n\in\mathbb{N}^{*})(\forall x\in\mathbb{R}^{*})(\exists y\in\mathbb{R}^{*})(\sqrt[n]{x}=y). \tag{2}$$

Which assert that, a hyperreal number has a hyperreal n-th root for all $n\in\mathbb{N}^{*}$. This the a $*$-transform of a sentence (1) which is also true.

Example. 4.1.2

There does not any numbers $x\in\mathbb{N}$. such that $x<1$.

This statement can formulated as follows:

$$\forall x\in\mathbb{N}(x<1) \tag{3}$$

Which is true. We can transform it to the true sentence:

$$\forall x\in\mathbb{N}^{*}(x<1) \tag{4}$$

Which assert that there are no number x of $\mathbb{N}^{*}$ smaller than 1.

From This example we conclude that a number of $\mathbb{N}^{*}/\mathbb{N}$ must be larger than all elements of $\mathbb{N}$, hence is infinitely large (unlimited).

Example 4.1.3

Archimedean property assert that, given any real numbers x there exists a natural number n (depending on x) such that $x<n$.

Archimedean property can expressing as follows:

$$(\forall x\in\mathbb{R})(\exists n\in\mathbb{N})(x<n) \tag{3}$$

Which is true. The $*$-transform of (3) can gives by:

$$(\forall x\in\mathbb{R}^{*})(\exists n\in\mathbb{N}^{*})(x<n) \tag{4}$$

Therefore since n is unlimited, then the statement (4) is true in $\mathbb{R}^{*}$.

The $*$-transform of terms can be defined by induction on the formation of τ as following laws:

- If τ is a constant or variable symbol then $\tau^{*}=\tau$.
- If $\tau=f(\tau_1,\ldots,\tau_n)$ then $=\tau^{*}=f^{*}(\tau_1^{*},\ldots,\tau_n^{*})$

A $*$-transform of a sentences can be defined as follows:

- Replacing each term τ occurring in Q by Q^{*}
- Replacing the relation symbol g of any atomic formula occurring in Q by g^{*}
- Replacing the symbol g of any quantifier $(\forall x\in P)$ or $(\exists x\in P)$ occurring in Q by g^{*}
- The symbols <,>; will denote the corresponding relation and functions in in $\mathbb{R}$ and $\mathbb{R}^{*}$ (Goldbring, 2014)

Thus, constructing a $*$-transform of a sentence L_θ really just consists of putting a $*$ on every term in Q, putting a $*$ on any relation symbol in Q and putting a $*$ on every set in Q acting as a bound on a variable (Davis, 2009; Keisler, 1976; Goldblatt, 2012). For writing first-order sentences, we can construct a method of transforming sentences in $\mathbb{R}$ to sentences in $\mathbb{R}^{*}$ as follows, the *-transform $\tau^{*}$of an L_θ-term τ which obtained by replacing each function symbol f occurring in τ by f^{*}. Therefore we define $*$-transforms of sentences explicitly on the construction of the sentence $\varnothing$ and ψ inductively as follows:

- $(\neg\varnothing)^{*}:=\neg(\varnothing)^{*}$
- $(\varnothing\wedge\psi)^{*}:=\varnothing^{*}\wedge\psi^{*}$
- $(\varnothing\vee\psi)^{*}:=\varnothing^{*}\vee\psi^{*}$
- $(\varnothing\rightarrow\psi)^{*}:=\varnothing^{*}\rightarrow\psi^{*}$
- $(\varnothing\leftrightarrow\psi)^{*}:=\varnothing^{*}\leftrightarrow\psi^{*}$
- $(\forall x\in Q)^{*}\varnothing:=(\forall x\in Q)^{*}\varnothing^{*}$
- $(\exists x\in Q)^{*}\varnothing:=(\exists x\in Q)^{*}\varnothing^{*}$

We now will an examples of first order L_θ-formula and its equivalence $*$-transform

Example. 4.1.4

The following first-order of L_θ-formula of the totally ordering field which mentioned in example (2.5.1) is equivalent to first order $L_{\theta^{*}}$ as follows:

- **Associativity Prosperity:**

$$(\forall x\in\mathbb{R}^{*})(\forall y\in\mathbb{R}^{*})(\forall z\in\mathbb{R}^{*})(x+(y+z)=(x+y)+z)$$
$$(\forall x\in\mathbb{R}^{*})(\forall y\in\mathbb{R}^{*})(\forall z\in\mathbb{R}^{*})(x\cdot(yz)=(xy)\cdot(z))$$

- **Commutative Prosperity**:

$$(\forall x\in\mathbb{R}^{*})(\forall y\in\mathbb{R}^{*})(x+y=y+x)$$
$$(\forall x\in\mathbb{R}^{*})(\forall y\in\mathbb{R}^{*})(x-y=y\cdot x)$$

- **Distributive Prosperity**:

$$(\forall x\in\mathbb{R}^{*})(\forall y\in\mathbb{R}^{*})(\forall z\in\mathbb{R}^{*})$$
$$(x\cdot(y+z)=x\cdot y+x\cdot z)$$

- **Existence of Identities:**

$$\left(\exists x \in \mathbb{R}^*\right)\left((0 = x) \wedge \left(\forall y \in \mathbb{R}^*\right)(x + y = y)\right)$$
$$\left(\exists x \in \mathbb{R}^*\right)\left((1 = x) \wedge \left(\forall y \in \mathbb{R}^*\right)(x \cdot y = y)\right)$$

- **Existence of Inverses:**

$$\left(\forall x \in \mathbb{R}^*\right)\left(\exists y \in \mathbb{R}^*\right)(x + y = 0)$$
$$\left(\forall x \in \mathbb{R}^*\right)\left(x \neq 0 \rightarrow \left(\exists y \in \mathbb{R}^*\right)(x \cdot y = 1)\right)$$

- **Total ordering Properity:**

$$\left(\forall x \in \mathbb{R}^*\right)\left(\forall y \in \mathbb{R}^*\right)\left((x \leq y) \wedge (y \leq x)\right) \rightarrow (x = y)\right)$$
$$\left(\forall x \in \mathbb{R}^*\right)\left(\forall y \in \mathbb{R}^*\right)\left(\forall z \in \mathbb{R}^*\right)\left(\left((x \leq y) \wedge (y \leq z)\right) \rightarrow (x \leq z)\right)$$
$$\left(yx \in \mathbb{R}^*\right)\left(\forall y \in \mathbb{R}^*\right)\left((x \leq y) \vee (y \leq x)\right)$$

So the list of first-order sentences L_{θ^*} above are true in $\mathbb{R}^*$.

Recall that the $*$-symbol can droped in the following case:

- If the symbol refer to the transforms of well-known relations such as =, ≠, <, ≥, ≤, ……etc.
- If the symbol referring to well-known mathematical functions such as sin, cos, tan, cot, …….. etc.
- If: $X \rightarrow Y$, then $f^*: X^* \rightarrow Y^*$ and $f^*(x) = f(x)$ if $x \in X$. Often the *- symbol in f^* may be dropped
- Consider addition in $\mathbb{R}$. Its $*$-transform is $*$-addition in $\mathbb{R}^*$ and $x{*}{+}y = x + y$ if $x, y \in \mathbb{R}$. The $*$-symbol can safely be dropped

Atomic relations are relations in which neither logical connectives nor quantifiers play a part, but only such relations as < or ∈, etc. Consider first < in $\mathbb{R}$, leading to $<^*$ in $\mathbb{R}^*$. Similarly as under e) we have that $x <^* y$ is equivalent to $x < y$ if $x, y \in \mathbb{R}$ and again the $*$-symbol can safely be dropped.

The logical connectives (∧, ∨, →, ↔, ¬) and both quantifiers (∀, ∃). For all of them the $*$-transform is identical to the inverse image, so that $*$-symbol should be dropped.

The idea of constructing a $*$-transform such that the first-order of sentence would be true if and only if L_{θ^*} is true, is called the transfer principle.

The Transfer Principle for First-Order Sentences

Theorem. 4.2.1. (Transfer Principle)

A sentence φ in $\mathbb{R}$ is true if and only if $*\varphi$ in $\mathbb{R}^*$ is true.

Proof

We will prove this by induction on sentences. For the base case, suppose the atomic formula $P(\tau_1,\ldots,\tau_n)$ is true for chosen values of $\tau_1,\ldots,\tau_n$. The $*$-transform of this sentence is:

$$P\left(\tau_1^*(j),\ldots\ldots.\tau_1^*(j)\right)$$

which is true if and only if the set of indices j is in ultrafilter. But the *-transforms of constants in $\mathbb{R}$ are just the corresponding constant sequences, so by the definition of τ^* it follows that the set of indices j such that:

$$P\left(\tau_1^*(j),\ldots\ldots.\tau_1^*(j)\right)$$

are in $\mathbb{N}$, which must be in our ultrafilter. Therefore:

$$P\left(\tau_1,\ldots,\tau_n\right) \rightarrow \left(P\left(\tau_1,\ldots,\tau_n\right)\right)^*.$$

Conversely, suppose $\neg P(\tau_1,\ldots,\tau_n)$. Then, by the same argument, set of indices j such that:

$$P\left(\tau_1^*(j),\ldots\ldots.\tau_1^*(j)\right)$$

is the empty set, which cannot be in our ultrafilter. Therefore:

$$\neg P\left(\tau_1,\ldots,\tau_n\right) \rightarrow \left(P\left(\tau_1,\ldots,\tau_n\right)\right)^*$$

hence:

$$P\left(\tau_1,\ldots,\tau_n\right) \leftrightarrow \left(P\left(\tau_1,\ldots,\tau_n\right)\right)^*$$

Hence the $*$-transforms of first-order sentences, which we know to be true by the transfer principle (Goldblatt, 1998).

Thus the Transfer Principle asserts that every first order statement true over $\mathbb{R}$ is similarly true over $\mathbb{R}^*$ and vice versa. This means that every statement is valid for areal $\mathbb{R}$ if and only if the corresponding formula is valid on a hyperreal $\mathbb{R}^*$, hence the transfer Principle asserts that every first order statement true over $\mathbb{R}$ is similarly true over $\mathbb{R}^*$ and vice versa (Davis, 2009). This means that the truth of the statements follows by the transfer principle from the fact that the sentence is true in its standard structure.

Now we will introduce some example.

Example. 4.2.1

The following first-order sentence which expressing the Archimedean property of the real numbers, using the mathematical logic it can be written as:

$$(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(x < n), \tag{5}$$

By applying a transfer principle the equivalent first-order of L_{θ^*}-sentence given by:

$$(\forall x \in \mathbb{R}^*)(\exists n \in \mathbb{N}^*)(x < n) \tag{6}$$

Example.4.2.2

Consider the standard mathematical functions which given by:

- $(\forall x \in \mathbb{R})\ \cos(\pi - x) = \cos x$
- $(\forall x \in \mathbb{R})\ \cosh x + \sinh x = e^x$
- $(\forall x, y \in \mathbb{R})\ \log xy = \log x + \log y$

is true. Then it can be transferrable to a hyperreal $\mathbb{R}^*$ hence the following facts:

- $(\forall x \in \mathbb{R}^*)\ \cos(\pi - x) = \cos x$
- $(\forall x \in \mathbb{R})\ \cosh x + \sinh x = e^x$
- $(\forall x, y \in \mathbb{R})\ \log xy = \log x + \log y$

Also are true.

Since areal number $\mathbb{R}$ is an ordered field which we expressed it in finite number of L_θ-sentence, by transfer principle we can conclude that the $*$-transform of these L_θ-sentence are true hence these explain that $\mathbb{R}^*$ is an ordered field. So, instead of explicitly proving the ordered field axioms ordered field axioms, we can simply take the $*$-transforms of list which we mentions in example (4.1.4) of first-order sentences of, that it is true by the transfer principle. We have thus proven that $\mathbb{R}^*$ is a totally ordered field without ever considering $\mathbb{R}^*$ as an ultrapower of $\mathbb{R}$, nor even doing a single ultrafilter calculation.

The Existential of Transfer Principle

The existential of transfer principle states that "If there exists a hyperreal number satisfying a certain property then there exists a real number with this property".

This principle can used to conclude that the original sequence must be bounded in areal $\mathbb{R}$.

Theorem. 4.3.1

If the extend hyperal sequence $u^*: \mathbb{N}^* \to \mathbb{R}^*, \mathbb{R}$ is never takes infinitely large values then the extend sequence u^* is bounded in $\mathbb{R}$.

Proof

Let u: $\mathbb{N} \to \mathbb{R}$ be a sequence, suppose c be an element of $(\mathbb{N}^* \setminus \mathbb{N})$, suppose that the sentence:

$$(\forall n \in \mathbb{N}^*)(|u^*(n| < c)$$

is true. Observe that these sentence is not $*$-transform of the L_θ-sentence because of the constant c. We can replace the constant c by existentially an quantified variable as L_θ-sentence as follows:

$$(\exists b \in \mathbb{R}^*)(\forall n \in \mathbb{N}^*)(|u^*(n| < b),$$

Thus the extensiale transfer yields that:

$$(\exists b \in \mathbb{R})(\forall n \in \mathbb{N})(|u(n| < b)$$

Which prove that the sequence is bounded in $\mathbb{R}$.

Proposition. 4.3.2

If f and g are functions of n variables on $\mathbb{R}$, then for $u = (u_1, \ldots, u_n)$ we have:

- $(f + g)^*(u) = f^*(u) + g^*(u)$ and
- $(f \cdot g)^*(u) = f^*(u) \cdot g^*(u)$, when $u \in \operatorname{dom} f^* \cap \operatorname{dom} g^*$
- $f^*(u)| = |f(u)|^*$, when $x \in \operatorname{dom} f^*$

Proposition. 4.3.3

If U and V are two sets in $\mathbb{R}^n$ then:

- $\varnothing^* = \varnothing$
- $(A \cup B)^* = U^* \cup V^*$
- $(U \cap V)^* = U^* \cap V^*$
- $(U')^* = (U^*)'$
- $U \subseteq V$ then $U^* \subseteq V^*$

which expresses the facts true in $\mathbb{R}^*$ (Goldblatt, 1998).

Now we present theorem which also assert that a first-order formula in $\mathbb{R}$ is true if and only if it is true in $\mathbb{R}^*$, so its the transfer principle is direct consequence of.

That is Lo's' theorem which is also sometimes known as the fundamental theorem of Ultra products. We give its formal statement below.

Theorem 4.3.4 (Lo's' Theorem)

For any L_θ formula $Q(u_1, u_2 \ldots u_m)$ and any $r^1, r^2 \ldots r^m \in \mathbb{R}^n$ the sentence $Q^*([r^1], \ldots, [r^m])$ is true if and only if $Q(r_n^1, \ldots, r_n^m)$ is true for almost all $n \in \mathbb{N}$. In other words:

$$Q^*([r^1], \ldots, [r^m]) \mathit{is\,true\,iff} \left[Q(r^1, \ldots, r^m)\right] \in F$$

The Lo´s' Theorem include transfer as special case because if Q is a sentence then it has no free variables so that $Q(v_1, \ldots, v_m)$ is just Q and likewise for Q^*. Thus $[Q(r^1, \ldots, r^m)]$ is in $\mathbb{N}$ if Q is true and $\varnothing$ otherwise, independently of sequences r^j. Since $\varnothing \notin$ Lo's' Theorem in this case simply says, Q^* is true if Q is true. Which is the transfer principle (Goldblatt, 1998; Staunton, 2013).

The transfer principle in this formulation expresses the fact that any classical statement is equivalent to the non-standard statement obtained by replacing everything by its $*$-transform except the bound variables in the statement (Keisler, 1976).

Theorem 4.3.5

For classical statement, $Q(u_1, u_2.....u_m)$ with a finite number of constants or free variables $v_1, ..., v_p$, then $Q(v_1, ..., v_m) \equiv Q\left(v_1^*, ..., v_m^*\right)$.

Proof

In Los' theorem take $v_1 = v_1^*$ ...... $v_m = v_m^*$ then $[Q(v_1, ..., v_m)]^* \equiv Q\left(v_1^*, ..., v_m^*\right)$, but $[Q(v_1, ..., v_m)]^* \equiv Q(v_1, ..., v_m)$.

One fact is disguised in this formulation of transfer, namely that in the statement to the right the bound variables need not be standard.

Now we come to the main result of our study. In the following section we apply the transfer principle to given a classical theory of sequences so as to obtain the nonstandard model equivalents (Hurd and Loeb, 1985).

Nonstandard models for Real Valued Convergence Sequences

In this section we will apply a transfer principle to a give a nonstandard model equivalents for the basic theory of convergence sequences. we will illustrate these results by considering the basic theory of limits, cluster point, Cauchy convergent criterion for real sequences.

Through this section we will use the fact that a hyperreal $\mathbb{R}^*$ is order field which has areal number $\mathbb{R}$ as a subfield and including unlimited numbers N is a subset of $\mathbb{N} \setminus \mathbb{N}^*$ there for infinitesimals and satisfies a transfer principle (Goldblatt, 2012).

Nonstandard Model for Convergence theorem.

Definition. 5.1.1

Let $\langle S_n: n\in \mathbb{N} \rangle$, $S: \mathbb{N} \to \mathbb{R}$ be standard sequence of areal number $\mathbb{R}$. Let $s(n) = s_n$, denote the sequence by $\langle s_n: n\in \mathbb{N} \rangle$ which converges to the limit $L\in \mathbb{R}$. Given $\varepsilon > 0$ in $\mathbb{R}$ there is a $k(\varepsilon)\in \mathbb{N}$ so that $|s_n-L| < \varepsilon$, for all $n>k$.

Formally standard condition of convergence sequence can expresses by the statement:

$$\left(\forall \varepsilon \in \mathbb{R}^+\right)\left(\exists k(\varepsilon) \in \mathbb{N}\right)\left(\forall x \in \mathbb{N}\right)\left(n > k(\varepsilon) \to \left|s_n - L\right| < \varepsilon\right).$$

Remark

- For $\varepsilon > 0$ the interval $(L-\varepsilon, L + \varepsilon)$ contains standard tails of sequence $\langle S_n: n\in \mathbb{N} \rangle$ i.e., contains the terms $s_n, s_{n+1}, s_{n+2}, \ldots\ldots$ From some point on it
- The sequence $S: \mathbb{N} \to \mathbb{R}$ has $*$-transform, $S^*: \mathbb{N}^* \to \mathbb{R}^*$ hence $s^*(n) = s_n^*$ for $n\in \mathbb{N}^*$

Theorem. 5.1.1

Areal value sequence $\langle s_n: n\in \mathbb{N} \rangle$ convergence to $L\in \mathbb{R}$, iff $s_n \simeq L$, for all unlimited n.

Proof

Suppose sequence $\langle s_n: n\in \mathbb{N} \rangle$ converges to L for $N\in \mathbb{N}_\infty^*$ is unlimited we want to show $s_n \simeq L$. Recall that the standard convergence condition implies that $\forall \varepsilon > 0$ there is $k(\varepsilon)\in \mathbb{N}$ so that:

$$\left(\forall n \in \mathbb{N}\right)\left(n > k(\varepsilon) \to \left|s_n - L\right| < \varepsilon\right)$$

is true in $\mathbb{R}$. Then by (universal) transfer:

$$\left(\forall n \in \mathbb{N}^*\right)\left(n > k(\varepsilon) \to \left|S_n^* - L\right| < \varepsilon\right)$$

And so for $n\in \mathbb{N}_\infty^*$ and $N > k(\varepsilon)$ then $\left|S_n^* - L\right| < \varepsilon$. Since n is unlimited and $k(\varepsilon)$ is limited so $\left|s_n^* - L\right| < \varepsilon$ for all unlimited $N\in \mathbb{N}_\infty^*$ then we conclude that $\forall \varepsilon > 0$, $\left|s_n^* - L\right| \simeq 0$ for all unlimited $n\in \mathbb{N}_n^*$, hence $s_n \simeq L$.

Conversely, suppose $s_n \simeq L$, for an unlimited $N\in \mathbb{N}_\infty^*$ then $\forall n\in N^*$ if, $n>N$, hence n is also unlimited therefore $|s_n-L|<\varepsilon$, so, $s_n \simeq L$. Let z be an element of $\in \mathbb{N}_\infty^*/N$, therefore by a hypothesis it is true:

$$\left(\forall n \in \mathbb{N}^*\right)\left(n > N \to \left|s_n - L\right| < \varepsilon\right)$$

Therefore:

$$\left(\exists z \in \mathbb{N}^*\right)\left(\forall n \in \mathbb{N}^*\right)\left(n > z \to \left|s_n - L\right| < \varepsilon\right)$$

is true. But a sentence (4.4) formatted existential transfer, yields:

$$\left(\exists z \in \mathbb{N}\right)\left(\forall n \in \mathbb{N}\right)\left(n > z \to \left|s_n - L\right| < \varepsilon\right)$$

is true. Which is the desired conclusion that a sequence $\langle s_n: n\in \mathbb{N} \rangle$ is converges.

Nonstandard Model for a Limits of a Convergence Sequences

Theorem. 5.2.1

Let $\langle s_n: n\in \mathbb{N} \rangle$ and $\langle s_n: n\in \mathbb{N} \rangle$ be standard sequences of real numbers. Then if $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} t_n = M$ in $\mathbb{R}$ then following properties are holds:

I. $\lim_{n\to\infty}(s_n + t_n) = L + M$

II. $\lim_{n\to\infty}(cs_n) = cL$, for any $c\in\mathbb{R}$

III. $\lim_{n\to\infty}(s_n t_n) = LM$

IV. $\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{L}{M}$, if $M \neq 0$

Proof

(I) we have $s_n \simeq L$ and $t_n \simeq M$ and hence $s_n + t_n \simeq L + M$ for any infinite n. The proof of II, III and IV are left to the reader.

Proposition. 5.2.1

Areal valued sequence $\langle s_n{:}n\in\mathbb{N}\rangle$ has at most one limits.

Proof

Suppose that a sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ converges to L and M then:

- $\lim_{n\to\infty} s_n = L$ or, $s_n \simeq L$
- $\lim_{n\to\infty} s_n = M$ or, $s_n \simeq M$

Therefore we have, $M \simeq s_n \simeq L$, so $L \simeq M$, hence $L = M$.

Definition. 5.2.1

The standard definition of a limit point L of a real value sequence$\langle s_n{:}\ n\in\mathbb{N}\rangle$ states that, for a given $\varepsilon > 0$ in $\mathbb{R}$ and $k\in\mathbb{N}$ there is $n\in\mathbb{N}$ and $n > k$, so that $|s_n\text{-}L|<\varepsilon$.

Nonstandard Model for Bounded Sequences

Definition 5.3.1. (The Standard Definitions of Bounded Sequences.)

A sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ is said to be bounded if there is $M\in\mathbb{N}$ so that the sentence:

$$(\forall n\in\mathbb{N})\left|s_n\right| < M$$

Theorem. 5.3.1

Areal valued sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ is abounded in $\mathbb{R}$ if and only if its extended terms are all limited.

Proof

By universal transfer the sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ the extended sequence $|s_n|<m$ is contained in $-M^* < s_n < M^*$ and therefore it is limited.

Conversely suppose $\langle s_n{:}\ n\in\mathbb{N}\rangle$ is limited for all infinite (or unlimited) $n\in\mathbb{N}^*_\infty$ then its limited for all $n\in\mathbb{N}^*$. Hens if r is any positive unlimited hyperreal then $|s_n|<r$ for all $n\in\mathbb{N}^*$ So by apply the transfer principle the sentence:

$$\left(\exists y\in\mathbb{R}^*\right)\left(\forall n\in\mathbb{N}^*\right)\left|s_n\right| < z$$

is true. By the existential transfer (4.3.1) it follow that there is some real number that is an upper to $|s_n|$ for all $n\in\mathbb{N}$ (Goldblatt, 2012).

Proposition 5.3.1

The sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ is bounded if and only if s_n^* is finite for all unlimited n.

Proof

Suppose $\langle s_n{:}\ n\in\mathbb{N}\rangle$ is bounded then there is a $k\in\mathbb{N}$ so that the sentence:

$$(\forall n)\left[N\langle n\rangle \rightarrow \left|s_n\right| \leq M\right]$$

is true in $\mathbb{R}$. By transfer, $s_n^* < k$ for all $n\in\mathbb{N}^*$, hence $|s_n^*|$ is finite for all unlimited n.

Conversely, if $\langle s_n{:}\ n\in\mathbb{N}\rangle$ is finite for all unlimited n. So by applying transfer the sentence:

$$\left(\exists z\in\mathbb{R}^*\right)\left(\forall n\in\mathbb{N}^*\right)\left|s_n\right| < z$$

Thus extential transfer (4.3.1) yields that:

$$(\forall n\in\mathbb{N})\left|s_n\right| \leq M$$

is true. Which prove that the sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ is bounded in $\mathbb{R}$.

Definition 5.3.1

Areal value sequence$\langle s_n{:}\ n\in\mathbb{N}\rangle$ is said to be:

- **Bounded above**. If there if there is a real upper bound to the sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ if and only if it has no positive unlimited extended terms
- **Bounded Below.** If there if there is a real upper bound to the sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ if and only if it has no negative unlimited extended terms

Nonstandard Model for the Cauchy Sequences

Definition. 5.4.1

Recall the standard condition for a sequence $\langle s_n{:}\ n\in\mathbb{N}\rangle$ to be a Cauchy sequence is Given $\varepsilon > 0$ in $\mathbb{R}$ there is $k\in\mathbb{N}$ such that:

$$\lim_{n\to\infty}\left|s_n - s_m\right| = 0.$$

m, $n \in \mathbb{N}$. Formally standard Cauchy sequence can expresses by the statement:

$$\left(\forall \varepsilon \in \mathbb{R}^{+}\right)(\exists k \in \mathbb{N})(\forall m,n \in \mathbb{N})\left(m,n \geq k \rightarrow \left|s_m - s_n\right| < \varepsilon\right)$$

Lemma. 5.4.1

The sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ is a Cauchy sequence iff $s_n^* \simeq s_m^*$, for all unlimited n and m.

Theorem. 5.4.2. (Cauchy Convergence Criterion)

A real valued sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ converges in $\mathbb{R}$ iff it is a Cauchy sequence.

Proof

The sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ is a Cauchy sequence then it is bounded. Suppose $m \in {}^*\mathbb{N}_\infty$ be unlimited number since s_m is limited then from theorem (5.4.1) hence s_m-$L \simeq 0$, i.e., (s_m is close to L in $\mathbb{R}$), because of the fact that all extended terms of a sequence are infinitely closed to each other we conclude that they infinitely close to $L \in \mathbb{R}$ hence $s_m \simeq L$. Hence by theorem (5.2.1) a sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ is converges to $L \in \mathbb{R}$. Conversely if $\langle s_n\colon n \in \mathbb{N} \rangle$ converges to L then $s_n^* \simeq L \simeq s_m^*$ for all unlimited n, m, then by theorem (5.1.1) and (5.4.1) a sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ is a Cauchy sequence.

Theorem. 5.4.3

The sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ is Cauchy if and only if for infinitely large values of n-the terms of the sequence are infinitesimally close (Davis, 2009).

Proof

Let $s : \mathbb{N} \rightarrow \mathbb{R}$ be a real-valued Cauchy sequence. That the property of Being Cauchy is first-order defined as:

$$\left(\forall \varepsilon \in \mathbb{R}^{+}\right)(\exists N \in \mathbb{N})(\forall m,n \in \mathbb{N})\left(m,n \geq N \rightarrow \left|s_n - s_m\right| < \varepsilon\right)$$

Since s is Cauchy this sentence is true, hence its $*$-transform:

$$\left(\forall \varepsilon \in \mathbb{R}^{*}\right)\left(\exists N \in \mathbb{N}^{*}\right)\left(\forall m,n \in \mathbb{N}^{*}\right)\left(m,n \geq N \rightarrow \left|s_n^* - s_m^*\right| < \varepsilon\right)$$

is also true. It applied to the extended hyperreal sequence $\left\langle s_n : n \in \mathbb{N}^*\right\rangle$. We know that there exists some $k \in N$ such that:

$$\forall n,m \in \mathbb{N}^*, m,n \geq, \left|s_n^* - s_m^*\right| < 1$$

Recall that a sequence is monotone if it is either an increasing or decreasing function. The following characterization is what would be expected, that for monotone sequences only one infinite number is needed for convergence.

Nonstandard Model for the a Monotonic Sequences

Definition. 5.5.1

Areal value sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ is said to be standard monotonic in $\mathbb{R}$ if it is either:

- Non decreasing, i.e., if $s_1 \leq s_2 \leq \cdots$
- Non increasing i.e., $s_1 \geq s_2 \geq \cdots$

Theorem. 5.5.1

A monotonic bounded sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ converges in $\mathbb{R}$ if it is:

I. Bonded above and no decreasing
II. Bounded below and no increasing

Proof

We shall proof case only *I*. Let $s_{\mathbb{N}}$ be extended term we show the term $s_{\mathbb{N}}$ has standard part L and also is a least upper bound of the set $\{s_n\colon n \in \mathbb{N}\}$ in $\mathbb{R}$. Then a statement $s_1 \leq s_2 \leq b$, holds for all $n \in \mathbb{N}^*$.

So it holds for all $n \in \mathbb{N}$ and by particular $s_1 \leq s_2 \leq b$, showing that $s_{\mathbb{N}}$ is limited, so indeed has standard part L.

Now we show that L is upper bound of the real. Since the sequence $\langle s_n\colon n \in \mathbb{N} \rangle$.

Is non-decreasing then by universal transfer we have. $n \leq m \rightarrow s_n \leq s_m$ for all n, $m \in \mathbb{N}^*$. In particular if $n \in \mathbb{N}$ then $n \leq \mathbb{N}$, so $s_n \leq s_N \simeq L$, giving $s_n \leq L$ as both number are real.

Now we will show L is the least upper bound in $\mathbb{R}$. For if t is any real upper bound of $\{s_n\colon n \in \mathbb{N}\}$, then by transfer $s_n \leq t$ for all $n \in \mathbb{N}^*$ so $L \simeq s_N \leq t$.

It follows that $L \leq t$ so L, $t \in \mathbb{R}$ (Goldblatt, 2012).

Nonstandard Model for a Cluster Point of a Sequences

Definition. 5.6.1. (Cluster point)

Let $\langle s_n\colon n \in \mathbb{N} \rangle$ be a real valued sequence, a point $L \in \mathbb{R}$ is a cluster point of a sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ if each interval $(L\text{-}\varepsilon, L\text{-}\varepsilon) \in \mathbb{R}$ contains many terms of a sequence.

We express a standard definition of a cluster point as the following sentences:

$$(\forall \varepsilon \in \mathbb{R})(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})\left(n > m \wedge \left|s_n - L\right| < \varepsilon\right)$$

Theorem. 5.6.1

A real valued sequence $\langle s_n\colon n \in \mathbb{N} \rangle$ has a point $L \in \mathbb{R}$ is a cluster point iff $s_N \simeq L$ for some unlimited N.

Proof

Suppose the sentence:

$$(\forall \varepsilon \in \mathbb{R})(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})\left(n > m \wedge \left|s_n - L\right| < \varepsilon\right)$$

Is hold. Let ε be appositive infinitesimal and $m \in \mathbb{N}_\infty^*$ then by transfer of a sentence:

$$\left(\forall \varepsilon \in \mathbb{R}^*\right)\left(\forall m \in \mathbb{N}^*\right)\left(\exists n \in \mathbb{N}^*\right)\left(n > m \wedge \left|s_n^* - L\right| < \varepsilon\right)$$

Hence there is some $n \in \mathbb{N}^*$ with $n > m$ and therefor n is unlimited and:

$$\left|s_n - L\right| < \varepsilon \simeq 0$$

Thus s_n is an extended term infinitely close to.

Now conversely suppose there is an unlimited N with $s_N \simeq L$. Let any positive integer $\varepsilon \in \mathbb{R}$ and $m \in \mathbb{N}$. Then $n > N > m$ and $|s_N - L| < \varepsilon$ this explain that:

$$\left(\exists n \in \mathbb{N}^*\right)\left(n > m \wedge \left|s_n - L\right| < \varepsilon\right)$$

Therefor by extential transfer $|s_n - L| < \varepsilon$ for some $n \in \mathbb{N}^*$, for all $n > m$.

Conclusion

A nonstandard analysis provides anew methodology for mathematics, in particularly for real analysis because the availability of infinitesimals allows for easier, directly and more inutility natural proofs in a hyperreal system of theorems which is holds in areal system.

The theorems we presented can be generalized to transferring a higher order sentences.

Recommendation for further research work to examine the nonstandard analysis for diver numbers systems.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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