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On the Relations among Characteristic Functions of Theta Functions

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Abstract: In this study, using the characteristic values $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{mod}\,2)$ a theorem on the $\frac{1}{2^r}$ coefficients of periods of first order theta function according to the $(1,\tau)$ period pair (for $r \in N^+$) is established. The following equalities are also obtained.

a) $$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}-\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

b) $$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{\pi i}{2^r}\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

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INTRODUCTION

Let $= SL_2(Z)$, we define $_N$ (or (N)) for each positive integer N to be subgroup of the modular group consisting of those matrices satisfying the condition $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\equiv I(\mathrm{mod}\,N)$

For unit matrix $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ in other words, $a\equiv d\equiv 1(\mathrm{mod}\,N)$ and $c\equiv b\equiv 0(\mathrm{Mod}N)$ [2].

We first define a theta characteristic to be a two by one matrix of integers, written $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. Next, given a complex number u and another complex number (Im > 0, $\Im$ to denote the upper half-plane). Z for the set of rational integers and (1) for the group. Let N 1 be an integer and put

$$\Gamma_0(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(1):\ c\equiv 0(\mathrm{Mod}N)\right\}.$$

Let be

$$U=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ V=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ W=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\ P=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}.$$

 $_U(2)$, $_V(2)$ and $_W(2)$ are defined by:

$_U(2) = \{S \in$ (1): $S \equiv I$ *or* $S \equiv U(Mod\ 2)\}$
$_V(2) = \{S \in$ (1): $S \equiv I$ *or* $S \equiv V(Mod\ 2)\}$
$_W(2) = \{S \in$ (1): $S \equiv I$ *or* $S \equiv W(Mod\ 2)\}$

were I is the unit matrix. The three subgroups $_U(2)$, $_V(2)$ and $_W(2)$ are conjugate. The subgroup θ of (1) is generated by U and V. For an odd positive integer n, the set of elements in θ of the from

$$\begin{pmatrix}a & b\\ nc & d\end{pmatrix}$$

is a subgroup of which will be denoted $\theta(n)$[3].

Definition1: For $u \in C$, $\tau \in \Im$ and characteristic value $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, the function defined as

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})\right\}\qquad(1)$$

is called first order theta function[1]

Definition2: A half-period is half of a period (in particular a complex vector), written

$$\begin{bmatrix}\mu\\ \mu'\end{bmatrix} \equiv \frac{1}{2}\begin{Bmatrix}\mu\\ \mu'\end{Bmatrix} = \frac{\mu'}{2} + \frac{\mu\tau}{2}.$$

A reduced half-period is half period in which μ and μ' equal 0 or 1 where μ and μ' are integers [1].
In the present study, whenever the integers μ and μ' will be as $\mu = 1$ and $\mu' = 1$, unless otherwise stated. In this study,

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix} (\bmod 2)$$

values of characteristic are used. When the periodicity of the function $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$ for $(1,\tau)$ period pair is examined.

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+1,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+1+\frac{\varepsilon'}{2})\right\}$$
$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) + n2\pi i + \pi i\varepsilon\right\}$$
$$= (-1)^{\varepsilon}\cdot\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$$

also

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau + \\ 2\pi i(n+\frac{\varepsilon}{2})(u+\tau+\frac{\varepsilon'}{2})\end{matrix}\right\}$$
$$= \sum_{n=-\infty}^{\infty} \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau + \\ 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) + n2\pi i\tau + \pi i\tau\varepsilon\end{matrix}\right\}$$
$$= (-1)^{\varepsilon'}.\exp(-\pi i\tau - 2\pi i u).\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau).$$

and

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau+1,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{\varepsilon}{2}) \\ (u+\tau+1+\frac{\varepsilon'}{2})\end{matrix}\right\}$$
$$= \sum_{n=-\infty}^{\infty} \exp\left\{\begin{matrix}\left(n+\frac{\varepsilon}{2}\right)^2\pi i\tau + 2\pi i\left(n+\frac{\varepsilon}{2}\right)\left(u+\frac{\varepsilon'}{2}\right) \\ + n2\pi i\tau + \pi i\tau\varepsilon\end{matrix}\right\}$$
$$= (-1)^{\varepsilon}.\exp(-\pi i\tau - 2\pi i u - \pi i\varepsilon').\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau).$$

By using $\eta_1 = (-1)^{\varepsilon}$, $\eta_2 = (-1)^{\varepsilon'}. \exp(-\pi i\tau - 2\pi iu)$ and $\eta_3 = (-1)^{\varepsilon}.\exp(-\pi i\tau - 2\pi iu - \pi i\varepsilon')$ we obtain

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau+1,\tau) = \eta_3.\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau).$$

As it is seen here, for $\eta_3 = 1$, because $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$ is doubly periodic, it would be an elliptic function.

Theorem

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau\right)$$
$$= \exp\left\{-\frac{1}{4^r}(\tau+2)\pi i - \frac{1}{2^r}(2u+\varepsilon')\pi i\right\}.\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$

where $r \in N^+$.

Proof

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)$$
$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{1}{2^r}+\frac{\tau}{2^r}++\frac{\varepsilon'}{2})\right\}$$
$$= \sum_{n=-\infty}^{\infty} \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) \\ + \frac{n\pi i\tau}{2^{r-1}} + \frac{\pi i\tau}{2^{r-1}} + \frac{\pi i\tau\varepsilon'}{2^r} + \frac{\pi i\varepsilon'}{2^r}\end{matrix}\right\}. \tag{2}$$

On the other hand, the reduced representative of an arbitrary characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ to be that reduced characteristic whose entries are the least nonnegative residues (mod2) of ε and ε'.
There are four reduced characteristic $\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$.
But $\theta\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)a \equiv 0.$

$$\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau) =$$
$$\sum_{n=-\infty}^{\infty} \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) + \frac{n\pi i\tau}{2^{r-1}} + \\ \frac{2\pi iu}{2^r} + \frac{2\pi i\varepsilon'}{2^r} + \frac{\pi i\tau\varepsilon}{2^r} + \frac{n\pi i}{2^{r-1}} + \frac{\pi i\tau}{4^r} + \frac{2\pi i}{4^r} + \frac{\pi i\varepsilon}{2^r}\end{matrix}\right\}$$

and

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i - \frac{1}{2^r}(2u+\varepsilon')\pi i\right\}.\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$
$$= \sum_{n=-\infty}^{\infty} \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) \\ + \frac{n\pi i\tau}{2^{r-1}} + \frac{\pi i\tau\varepsilon}{2^r} + \frac{n\pi i}{2^{r-1}} + \frac{\pi i\varepsilon}{2^r}\end{matrix}\right\} \tag{3}$$

By the theorem given above we can obtain the following characteristic equalities for u = 0 value of the complex variable

a) $\theta\begin{bmatrix}1\\1\end{bmatrix}\left(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau\right)$

$$=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{\begin{matrix}(n+\frac{1}{2}+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2}+\frac{1}{2^r}).\\(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}+\frac{\pi i}{2^{r-1}}+\frac{\pi i}{2^r}\end{matrix}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2})^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{\pi i\tau}{2^r}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i}{2^r}+\frac{\pi i}{2}+n\pi i\right\} \quad (4)$$

and

$$\theta\begin{bmatrix}1\\0\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{\begin{matrix}\left(n+\frac{1}{2}+\frac{1}{2^r}\right)^2\pi i\tau+2\pi i\left(n+\frac{1}{2}+\frac{1}{2^r}\right)\\\left(0+\frac{1}{2}+\frac{1}{2^r}\right)-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{r-1}}\end{matrix}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{\begin{matrix}(n+\frac{1}{2})^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{\pi i\tau}{2^r}\\+\frac{n\pi i}{2^{r-1}}+\frac{\pi i}{2^r}+\frac{\pi i}{2}+n\pi i\end{matrix}\right\}. \quad (5)$$

From the equations (4) and (5), we can get the following equality

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

b) $\theta\begin{bmatrix}0\\0\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2^r})(0+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}+\frac{\pi i}{2^{r-1}}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{n^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{n\pi i}{2^{r-1}}\right\} \quad (6)$$

and

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{\pi i}{2^r}\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2^r})(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{r-1}}-\frac{\pi i}{2^r}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\exp\left\{n^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+n\pi i+\frac{n\pi i}{2^{r-1}}\right\} \quad (7)$$

If n = 2k ∈ Z, then from the equalities (6) and (7) the following is obtained

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

$$=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

With the help of this theorem proved, transformations among theta functions can be found for characteristic value $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}$ according to all multiples $\frac{1}{2^r}$ of the periods.

The subject that should be discussed here is ; characteristic values

$$\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\text{mod } 2)$$

of first order theta function can be expressed as characteristic values

$$\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\\varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}.$$

This situation has proved that theta functions are generalized as characteristic values

$$\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\\varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}.$$

With the help of this alternative formula above, we can get the following equalities according to quarter-periods.

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix}(\text{mod } 2)$ the

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

$$=\sum_n \exp\left\{(n+\frac{1}{2})^2\pi i\tau+2\pi i(n+\frac{1}{2})(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{2}\right\}$$

$$=ie^{-\frac{\pi i\tau}{4}}\sum_n(-1)^n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\0\end{bmatrix}(\bmod 2)$ then

$$\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

$$=\sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{1}{2})(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}$$

$$=e^{-\frac{\pi i\tau}{4}}\sum_n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\} \quad (9)$$

Using the equations (8) and (9) we can get

$$\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=$$

$$\frac{ie^{-\frac{\pi i\tau}{4}}\sum_n(-1)^n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}}{e^{-\frac{\pi i\tau}{4}}\sum_n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}}$$

i). If n is 0 or even integer then,

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

ii). If n is odd integer then

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=-\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)\cdot$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\1\end{bmatrix}(\bmod 2)$ then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{2}\right\}=\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\0\end{bmatrix}(\bmod 2)$ then

$$(8)$$

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}=\sum_n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}$$

From the equations (10) and (11) we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=\frac{\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}{\sum_n\exp\left\{n^2\pi i\tau(2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}$$

iii). If n is 0 or even integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)\cdot$$

iv). If n is odd integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

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