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On the Elliptic Function Arising from the Theta Functions and Dedekind's η-Function

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Abstract: It's a reality that there is a relationship between the sigma function of Weierstrass and theta functions. An elliptic function can be set up using the theta functions just as it can be astablished with the help of sigma function of Weierstrass and two relations between the Dedekind's η-function and θ-theta function were established by the using characteristic values $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$ (mod2) for θ-function according to the (u,τ) pair and u,τ complex numbers, satisfying Im $\tau>0$. In this study, the transformations among the theta functions according to the quarter periods have been given and a Jacobian style elliptic functions has been set up the theta function by the help of a defined function.

Keywords: Dedekind function and Theta Functions, Characteristic Values

INTRODUCTION

The expansions, in infinite series, of the $\wp(u)$, (u) and (u)-functions of Weiestrass which we have so far considered, are not best suited to numerical computation.It is of advantage therefore to introduce another function, defined by $\theta(u,\tau)$, which has a rapidly convergent expansion in infinite series and which is directly connected whith the sigma function of Weierstrass. Let τ be a complex variable, with $\tau = \frac{\omega_2}{\omega_1}$ real, Im $\tau>0$
and $\omega = m\omega_1 + n\omega_2$ with m,n = 0,±1,±2,... and u a complex variable. We define the function $\theta(u,\tau)$ by the series,

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)=\sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})\right\} \quad (1)$$

where $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{mod}\,2)$, ε and ε' are integers and n ranges over all the integers ($-\infty$ to ∞)[1].
The series (1) converges absolutely and uniformly in compact sets of the u-complex plane and therefore represents an entire function of u .
We can see the following alternative theta functions

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\mu\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix} \quad (2)$$

where $\mu=\exp\{-\frac{1}{4^r}(\tau + 2)\, i - \frac{1}{2^r}(2u+\varepsilon')\, i\}$ and $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{mod}\,2)$ and ε' are integers

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)=\sum_n \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau+\\ 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})\end{matrix}\right\} \quad (3)$$

where $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{mod}\,2)$, ε and ε' are integers n ranges over all the integers ($-\infty$ to ∞) [2].

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)=\sum_n \exp\left\{\begin{matrix}(n+\frac{\varepsilon}{2})^2\pi i\tau+\\ 2i(n+\frac{\varepsilon}{2})(u-\frac{\varepsilon'}{2}\pi)\end{matrix}\right\} \quad (4)$$

where $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{mod}\,2)$, ε and $\varepsilon'$are integers n ranges over all the integers ($-\infty$ to ∞)

If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix}(\mathrm{mod}\,2)$ then

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau) = -i\sum_n (-1)^n \exp\begin{Bmatrix}(n+\frac{1}{2})^2\pi i\tau + \\ (2n+1)\pi iu\end{Bmatrix} \quad (5)$$

This the function $\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$ is alternative formula in [3]

If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}0\\0\end{bmatrix} (\bmod 2)$ and u = 0 then

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) = \sum_n \exp(n^2\pi i\tau) \quad (6)$$

This is the function $\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$ that is an alternative formula in [4].

Definition: A period, denoted $\begin{Bmatrix}a\\b\end{Bmatrix}$, is b+a τ. A quarter period is quarter of a period, written

$$\frac{1}{4}\begin{Bmatrix}a\\b\end{Bmatrix} = \frac{b}{4} + \frac{a\tau}{4}.$$

A reduced quarter-period is a quarter-period in which a and b equal 0 or 1 [5].
With the help of this alternative formula (2) above, we can get the following equalities according to quarter-periods.

If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\1\end{bmatrix} (\bmod 2)$ the

$$\theta\begin{bmatrix}1\\1\end{bmatrix}\left(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau\right) = \sum_n \exp\left\{(n+\frac{1}{2})^2\pi i\tau + 2\pi i(n+\frac{1}{2})(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{1}{2}\right\}$$
$$= ie^{-\frac{\pi i\tau}{4}}\sum_n (-1)^n \exp\left\{(n+\frac{1}{2})^2\pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{2} + \frac{n\pi i\tau}{2} + \frac{\pi i\tau}{4} + \frac{\pi i}{4}\right\} \quad (7)$$

If $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\0\end{bmatrix} (\bmod 2)$ then

$$\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau) = \sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau + 2\pi i(n+\frac{1}{2})(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}$$
$$= e^{-\frac{\pi i\tau}{4}}\sum_n \exp\left\{(n+\frac{1}{2})^2\pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{2} + \frac{n\pi i\tau}{2} + \frac{\pi i\tau}{4} + \frac{\pi i}{4}\right\} \quad (8)$$

Using the equations (7) and (8) we can get

$$\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)} = \frac{ie^{-\frac{\pi i\tau}{4}}\sum_n (-1)^n \exp\left\{(n+\frac{1}{2})^2\pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{2} + \frac{n\pi i\tau}{2} + \frac{\pi i\tau}{4} + \frac{\pi i}{4}\right\}}{e^{-\frac{\pi i\tau}{4}}\sum_n \exp\left\{(n+\frac{1}{2})^2\pi i\tau + (2n+1)\pi iu + \frac{n\pi i}{2} + \frac{n\pi i\tau}{2} + \frac{\pi i\tau}{4} + \frac{\pi i}{4}\right\}}$$

a. If n is 0 or even integer then,

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau) = i\,\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

b. If n is odd integer then

$$\theta\begin{bmatrix}1\\1\end{bmatrix}\left(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau\right)=-i\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\1\end{bmatrix}(\bmod 2)$ then

$$\begin{aligned}\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n \exp\left\{n^2\pi i\tau+2n\pi i(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{2}\right\}\\&=\sum_n(-1)^n\exp\begin{Bmatrix}n^2\pi i\tau+2n\pi iu\\+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\end{Bmatrix}\end{aligned}\qquad(9)$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\0\end{bmatrix}(\bmod 2)$ then

$$\begin{aligned}\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n \exp\left\{n^2\pi i\tau+2n\pi i(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}\\&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}\end{aligned}\qquad(10)$$

From the equations (9) and (10) we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=\frac{\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}{\sum_n\exp\left\{n^2\pi i\tau(2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}$$

c. If n is 0 or even integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

d. If n is odd integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=-\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

Theorem 2: The function $\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$ defined in [4] is odd function of u and it can be expressed by infinite product

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)=ce^{\frac{\pi i\tau}{4}}2\sin\pi u\prod_{n=1}^{\infty}\left\{1-e^{2(n\tau+u)\pi i}\right\}\prod_{n=1}^{\infty}\left\{1-e^{2(n\tau-u)\pi i}\right\}$$

where $c=\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau})$, $\operatorname{Im}\tau>0$. We consider the function $\varphi(u,\tau)$ expressed by product

$$\varphi(u,\tau)=\prod_{n=1}^{\infty}\left\{1-e^{[(2n-1)\tau+2u]\pi i}\right\}\prod_{n=1}^{\infty}\left\{1-e^{[(2n-1)\tau-2u]\pi i}\right\}$$

Theorem 3: The function $\dfrac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}$ is an elliptic function with periods 1 and τ.

Proof: Let $\psi(u,\tau)=\dfrac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}$

$$\psi(u+1,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+1,\tau)}{\varphi(u+1,\tau)}=\frac{-\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(-u,\tau)}{\varphi(u,\tau)}$$

where $\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)=-\theta\begin{bmatrix}1\\1\end{bmatrix}(-u,\tau)$ from theorem 2.

$$\psi(u+\tau,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\tau,\tau)}{\varphi(u+\tau,\tau)}=\frac{-e^{-(2u+\tau)\pi i}\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{e^{-(2u+\tau)\pi i}\varphi(u,\tau)}=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}$$

since

$$\varphi(u+\tau,\tau)=\prod_{n=1}^{\infty}\left\{1-e^{(2n-1)\tau\pi i+2\pi i(u+\tau)}\right\}\prod_{n=1}^{\infty}\left\{1-e^{(2n-1)\tau\pi i-2\pi i(u+\tau)}\right\}$$

$$=\prod_{n=1}^{\infty}\left\{1-e^{(2n+1)\tau\pi i+2\pi iu}\right\}\prod_{n=1}^{\infty}\left\{1-e^{(2n-3)\tau\pi i-2\pi iu}\right\}$$

$$=\prod_{n=1}^{\infty}\left\{1-e^{[2(n+1)-1]\tau\pi i+2\pi iu}\right\}\prod_{n=1}^{\infty}\left\{1-e^{[2(n-1)-1]\tau\pi i-2\pi iu}\right\}$$

$$=\prod_{m=2}^{\infty}\left\{1-e^{(2m-1)\tau\pi i+2\pi iu}\right\}\prod_{n=0}^{\infty}\left\{1-e^{(2m-1)\tau\pi i-2\pi iu}\right\}$$

$$=\prod_{m=1}^{\infty}\left\{1-e^{(2m-1)\tau\pi i+2\pi iu}\right\}\prod_{m=1}^{\infty}\left\{1-e^{(2m-1)\tau\pi i-2\pi iu}\right\}\left\{1-e^{-(\pi i\tau+2\pi iu)}\right\}\left\{1-e^{(\pi i\tau+2\pi iu)}\right\}^{-1}$$

$$=-e^{-(2u+\tau)\pi i}\varphi(u,\tau)$$

where n = m. The function $\psi(u,\tau)$ is therefore a doubly periodic with periods 1 and τ having neither zeros nor poles on account of the fact that $\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$ possesses the same periodicity factors as φ (u,τ). Hence the function $\psi(u,\tau)$ is an elliptic function since the set of all meromorphic functions form a field and $\psi(u,\tau)$ is meromorphic and periodic with periods 1 and τ. The function (u,τ) of Weierstrass, defined by the series

$$\zeta(u,w)\equiv\zeta(u;\omega_1,\omega_2)=\frac{1}{u}+\sum_{w\neq 0}\left[\frac{1}{u-w}+\frac{1}{w}+\frac{u}{w^2}\right]$$

is not elliptic since it is not double periodic. Hence we write the Weierstrass's σ-sigma function, which has zero of order 1 at all lattice points, by the Weierstrass's infinite product

$$\sigma(u)\equiv\sigma(u;w)\equiv\sigma(u;\omega_1,\omega_2)=u\prod_{w\neq 0}\left(1-\frac{u}{w}\right)e^{\frac{u}{w}+\left(\frac{u}{2w}\right)^2}$$

Taking the logarithmic derivative formally yields the function $\zeta(u,w)=\frac{d}{du}Log\,\sigma(u,w)$ which is homogeneous of degree -1, namely $\zeta(\lambda u,\lambda w)=\lambda^{-1}\zeta(u,w)$ and it is not elliptic since it is not double periodic [3].

The modular group denoted by

$$\Gamma=\left\{\begin{bmatrix}a&b\\c&d\end{bmatrix}:a,b,c,d\in Z,\det\begin{bmatrix}a&b\\c&d\end{bmatrix}=ad-bc=1\right\}$$

is the multiplicative group of SL(2, Z) matrices with determinant 1.The matrices of $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ and $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ generate Γ. We first define the Riemann's θ-function by the series

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+a)^2\pi i\tau+2\pi i(n+a)(u+b)\right\}$$

with a given complex number u and complex number τ satisfying $\mathrm{Im}(\tau=\frac{\omega_1}{\omega_2}\neq real)>0$ and characteristic $\begin{bmatrix}a\\b\end{bmatrix}$ where a and b are rational numbers .

Let us recall the transformation

$$\theta\begin{bmatrix}\mu_1\\\mu_2\end{bmatrix}(u,\tilde{\tau})=K(M,\delta)\sqrt{c\tau+d}.\exp\left(\frac{\pi iu^2}{c\tau+d}\right).\theta\begin{bmatrix}\mu\\\mu'\end{bmatrix}(z,\tau)$$

where M= $\begin{bmatrix} q & b \\ c & d \end{bmatrix} \in SL(2, Z)$, $u = \dfrac{z}{c\tau + d}$, $\breve{\tau} = \dfrac{a\tau + b}{c\tau + d}$, $\mu_1 = d\mu - c\mu' - \dfrac{cd}{2}$ and $\mu_2 = -b\mu + a\mu' + \dfrac{ab}{2}$ and $K(M, \delta)$ is a certain root of 1. We want to define an action of an element of $M = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ on theta function. We try the special value of $M = \begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and the get

$$\theta\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}(u, \breve{\tau}) = \theta\begin{bmatrix} d\mu - c\mu' + \dfrac{cd}{2} \\ -b\mu + a\mu' + \dfrac{ab}{2} \end{bmatrix}\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \theta\begin{bmatrix} \mu \\ -\mu + \mu' + \dfrac{1}{2} \end{bmatrix}(z, \tau + 1)$$

$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\mu)^2 \pi i(\tau + 1) + 2\pi i(n+\mu)(z - \mu + \mu' + \frac{1}{2})\right\}$$

$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\mu)^2 \pi i\tau + 2\pi i(n+\mu)(z + \mu') + (n+\mu)^2 \pi i + 2\pi i(n+\mu)(-\mu + \frac{1}{2})\right\}$$

$$= \sum_{n=-\infty}^{\infty} \exp\left\{(n+\mu)^2 \pi i\tau + 2\pi i(n+\mu)(z + \mu') + \pi i(n^2 + n + \mu - \mu^2)\right\}$$

$$= \exp \pi i(\mu - \mu^2).\theta\begin{bmatrix} \mu \\ \mu' \end{bmatrix}(z, \tau)$$

since $n^2 + n \equiv n(n + 1)$ is congruent to zero module 2 [$n(n + 1)$ if $n \equiv 0 (\mathrm{mod}\ 2)$ and $\mathrm{l}n(n + 1)$] from $n + 1 \equiv 0(\mathrm{mod}\ 2)$ if $n \equiv 1(\mathrm{mod}\ 2)$)

In this study, we treat the special value of θ- function with characteristic

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u, \tau) = \sum_{n=-\infty}^{\infty} \exp\left\{ \begin{matrix} (n + \frac{\varepsilon}{2})^2 \pi i\tau + \\ 2\pi i(n + \frac{\varepsilon}{2})(u + \frac{\varepsilon'}{2}) \end{matrix} \right\}$$

where $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\mathrm{mod}\ 2)$ and $\varepsilon, \varepsilon'$ are integers.

Thus, we have the following relations

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u, \tau) = \sum_{n=-\infty}^{\infty} (\exp(n^2 \pi i\tau + 2n\pi iu).$$

$$\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(u, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(n^2 \pi i\tau + 2n\pi iu).$$

Theorem.4: The Weierstrass's σ-function is a theta function so that the connection between the σ-function and θ-function is established

$$\sigma(u; \omega_1, \omega_2) = \theta(\tfrac{u}{\omega_1}, \tau).\frac{\omega_1}{\theta'(0, \tau)}.e^{\frac{u^2}{\omega_1}\eta_1}$$

where $\eta_1 = \zeta(\frac{\omega_1}{2})$ and (ω_1, ω_2) is a pair of periods of the Weierstrass's elliptic function $\wp(u; \omega_1, \omega_2)$ [4]. Now, let us observe that

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u, \breve{\tau}) = \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u, \tau) = \sum_n \exp(n^2 \pi i\tau + 2n\pi iu)$$

Then we see that the function $\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)$ defined by the series in [1] is a alternative formula of $\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u, \tau)$.

The formulas $\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u, \tau)$ and $\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(u, \tau)$ were used in this article where $u \neq 0$.

At first we see the infinite products

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u, \tau) = \prod_{n=1}^{\infty}(1 - e^{2n\pi i\tau}).\prod_{n=1}^{\infty}(1 + e^{(2n-1)\pi i\tau + 2\pi iu}).$$

$$\prod_{n=1}^{\infty}(1 + e^{(2n-1)\pi i\tau - 2\pi iu}).$$

which it converges absolutely .

Theorem 5: We have the relations

a) $\eta(u) = e^{\frac{\pi i u}{12}}.\theta\begin{bmatrix}0\\0\end{bmatrix}(\frac{u+1}{2}, 3u+2k)$

b) $\theta\begin{bmatrix}0\\1\end{bmatrix}(\frac{u+4}{4}, \frac{3}{2}u) = e^{-\frac{\pi i u}{12}}\eta(u).\prod_{n=1}^{\infty}(1-e^{(2n-1)\pi i u})$

between the functions $\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)$, $\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau)$ and Dedekind's η-function which defined by the infinite product

$$\eta(u) = e^{\frac{\pi i u}{12}}.\prod_{n=1}^{\infty}(1-e^{2n\pi i u})$$

where Im$\tau > 0$ and k is a integer.

Proof: a) Let us recall the formula

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau) = \prod_{n=1}^{\infty}(1-e^{2n\pi i\tau}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau+2\pi i u}).$$
$$\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau-2\pi i u}).$$

If k integer, then we have

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2}, 3u+2k) = \prod_{n=1}^{\infty}(1-e^{2n\pi i(3u+2k)}).$$
$$\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)+2\pi i(\frac{u+1}{2})}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)-2\pi i(\frac{u+1}{2})})$$
$$=\prod_{n=1}^{\infty}(1-e^{6n\pi i u}).\prod_{n=1}^{\infty}(1+e^{6n\pi i u-2\pi i u-(2k-1)\pi i}).$$
$$\prod_{n=1}^{\infty}(1+e^{6n\pi i u-4\pi i u-(2k+1)\pi i}) = \prod_{n=1}^{\infty}(1-e^{6n\pi i u}).\prod_{n=1}^{\infty}(1-e^{6n\pi i u-2\pi i u}).$$
$$\prod_{n=1}^{\infty}(1-e^{6n\pi i u-4\pi i u})$$

If we set R= $e^{2\pi i u}$, then we obtain

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2}, 3u+2k) = \prod_{n=1}^{\infty}(1-R^{3n}).$$
$$\prod_{n=1}^{\infty}(1-R^{3n-1}).\prod_{n=1}^{\infty}(1-R^{3n-2})$$

On the other hand, we may set $n = n' + 1$, then

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2}, 3u+2k) = \prod_{n'=1}^{\infty}(1-R^{3n'+3}).$$
$$\prod_{n'=1}^{\infty}(1-R^{3n'+2}).\prod_{n'=1}^{\infty}(1-R^{3n'+1})$$
$$=(1-R)(1-R^2)(1-R^3)(1-R^4)\ldots=$$
$$\prod_{m=1}^{\infty}(1-R^m) = \prod_{m=1}^{\infty}(1-e^{2m\pi i u})$$

According to above, we have $\eta(u) = e^{\frac{\pi i u}{12}}.\theta\begin{bmatrix}0\\0\end{bmatrix}(\frac{u+1}{2}, 3u+2k)$ from the Dedekind's η-function defined by the infinite product $\eta(u) = e^{\frac{\pi i u}{12}}.\prod_{n=1}^{\infty}(1-e^{2n\pi i u})$ where m = n'.

b) According to the equation, $\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau) = \sum_{n=-\infty}^{\infty}(-1)^n\exp(n^2\pi i\tau + 2n\pi i u)$ we have

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4}, \tfrac{3}{2}u) = \sum_{n=-\infty}^{\infty}(-1)^n\exp\left[\tfrac{1}{2}n(3n+1)\pi i u\right]$$
$$=1+\sum_{n=1}^{\infty}(-1)^n\left\{\exp\left[\tfrac{1}{2}n(3n-1)\pi i u\right]+\exp\left[\tfrac{1}{2}n(3n+1)\pi i u\right]\right\}$$
$$=1+\sum_{n=1}^{\infty}(-1)^n\begin{bmatrix}x^{\frac{1}{2}n(3n-1)}+\\x^{\frac{1}{2}n(3n+1)}\end{bmatrix} = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+$$
$$\ldots+\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4}, \tfrac{3}{2}u) = (1-x)(1-x^2)(1-x^3)\ldots = \prod_{n=1}^{\infty}(1-x^n)$$

where $x = e^{\pi i u}$ for $|x| < 1$ and $\frac{1}{2}n(3n+1)$ are known as the pentagonal numbers n = -1,-2,-3,...

This results play a role of key stone in the forthcoming work concerning relation between the θ-theta function and Dedekind's η-function. In fact, if the application of theorem.(2-b)on the relation obtained with the theorem.(2-a) which known as the equation between Dedekind's η-function and L.Euler's theorem on pentagonal numbers is done, we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(\frac{u+4}{4}, \frac{3}{2}u)}{\prod_{n=1}^{\infty}(1-e^{(2n-1)\pi i u})} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n e^{(\frac{1}{2}n(3n+1)\pi i u)}}{\prod_{n=1}^{\infty}(1-e^{(2n-1)\pi i u})} =$$
$$\frac{\prod_{n=1}^{\infty}(1-e^{n\pi i u})}{\prod_{n=1}^{\infty}(1-e^{(2n-1)\pi i u})} = \prod_{n=1}^{\infty}(1-e^{2n\pi i u}) = e^{-\frac{\pi i u}{12}}.\eta(u)$$

As a result, the relation has been obtained between theta and Dedkind's $-\eta(u)$ functions by using the characteristic $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the variable $\frac{u+4}{4}$ instead of the characteristic $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the variable $\frac{u+1}{2}$ which were previously used by Jaccobi.

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