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On Asymptotics of Lp Extremal Polynomials on a Complex Curve Plus an Infinite Number of Points

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Abstract: We study, for all p>0 and under certain conditions the asymptotic behavior of Lp extremal polynomials with respect to the measure $\alpha=\beta+\gamma$. Where, β denotes a positive Szegö measure on the closed rectifiable Jordan curve E in the complex plane and γ has an infinity of masses in the region exterior to the curve E.

Key words: Asymptotic behavior, Lp extremal polynomials.

INTRODUCTION

Let F be a compact set in the complex plane C and α be a finite measure defined on the borel sets of C with $F = \text{support}(\alpha)$. We denote by $m_{n,p}(\alpha,F)$, $n\in N$, $p>0$ the extremal constants $m_{n,p}(\alpha,F) = \min\{\|Q_n\|_{Lp(\alpha,F)} : Q_n = z^n + a_{n-1}z^{n-1} + \ldots + a_0,\ a_0,\ldots,a_{n-1}\in C\}$ and by $T_{n,p}(\alpha,F)$ the associated extremal polynomials (we suppose that $z^n\in L_p(\alpha,F)$, $n\in N$). The case p = 2 is the special case of $T_{n,2}(\alpha,F)$ monic orthogonal polynomials.

There are many interesting problems about orthogonal or extremal polynomials. The most important and difficult ones are their asymptotic behavior and zero distributions.

The study of the asymptotic behavior of orthogonal or extremal polynomials contributed in the resolution of other important problems in Mathematics. We especially mention:

* The convergence of Padé approximants ($F = [1,+1] \cup \{z_k\}$, $p = 2$,[1])
* The spectral theory ($F=[-1,+1] \cup \{z_k\}$, $p = 2$,[2,3])
* The zero distribution of extremal polynomials ($F = \Gamma = \{z : |z| =1\}$, $p\geq 1$[4], $F = \Gamma \cup \{z_k\}$, $p = 2$,[5])
* The theory of the representation of analytic functions by series of polynomials ($F= \Gamma$ or $F = E$, E being a smooth Jordan curve,[6-8] .

If we are interested in asymptotics of extremal constants $m_{n,p}(\alpha,F)$ and $T_{n,p}(\alpha,F)$ polynomials, then the cases studied are the following :

1. $F = [-1,+1]$, $d\alpha(x) = \rho(x)dx$, $\rho(x)$ is non negative and integrable. For p = 2, we have the classical results of Szegö[8,9]. For $0<p<\infty$, $\rho(x) = t(x)/\sqrt{(1-x^2)}$ and log t(x) a Riemann integrable function, Berstein[10] found the power asymptotic of the extremal constants $m_{n,p}(\alpha,F)$. Lubinsky and Saff[11] generalized the result of Bernstein by considering $1/\rho(x) \in L_r[-1,+1]$, $r > 1$.
2. $F = E$, E is a smooth closed Jordan Curve and is absolutely continuous and satisfies the Szegö condition. The case $0<p<\infty$ was studied by Geronimus[12]. The special case of the unit circle and p=2 has a long history of study (see, for example,[6,8,9]).
3. $F = \cup^{\ell}_{k=1} E_k$, E_k being a smooth closed Jordan curve. This case was investigated by Widom[13].

For other studies on Lp extremal polynomials[14-18]. In this work we study the strong asymptotics of $m_{n,p}(\alpha,F)$ and $T_{n,p}(\alpha,F)$ in the case where $0<p<\infty$, $F = E \cup \{z_k\}^{\infty}_{k=1}\}$, E being a closed rectifiable Jordan curve with some smoothness conditions, $z_k \in \text{Ext}(E)$, $\alpha = \beta + \gamma$, with $\text{support}(\beta) = E$, $d\alpha = \rho(\xi)|d\xi|$ on E and $\gamma = \sum^{\infty}_{k=1} A_k\delta_{zk}$ (δ_{zk} being the Dirac delta unit measure supported at the point z_k).

This work is a generalization, on one hand[19] in which Kaliaguine uses a measure concentrated on E plus a finite number of points $\{z_k\}^{\ell}_{k=1}$ and on the other hand[20] where Khaldi studies the same problem, but in the case of a measure concentrated on E plus a infinite number of points $\{z_k\}^{\infty}_{k=1}$ with $p\geq 1$. In this study, all proofs contain more details and we particularly focused on the case $0<p<1$. The passage from a finite number to an infinite number of points is a difficult problem and its resolution required, in the case p=2, several years[21-23].

EXTREMAL PROBLEMS IN THE $H^p(Y,\rho)$ SPACES

Let E be a closed rectifiable Jordan curve, $Y = \text{Ext}(E)$, $G = \{w \in C : |w| > 1\}$, ($\infty \in Y$, $\infty \in G$) and $w = \Phi(z)$ is the function that maps Y conformally on G in such a manner that $\Phi(\infty) = \infty$ and $\lim_{z\to\infty}(\Phi(z)/z) > 0$. Really, this limit is equal to $1/C(E)$ where $C(E)$ is the logarithmic capacity of E. Let ψ be the inverse function of Φ , $\psi : G\to Y$. The two functions $\Phi(z)$ and $\psi(w)$ have a continuous extension to E and on the unit circle, respectively. Let $\rho(\xi)$ be an integrable non negative function on E. If the weight function $\rho(\xi)$ satisfies the Szegö condition:

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$$\int_E (\log \rho(\xi)) \,|\Phi'(\xi)|\, |d\zeta| > -\infty, \quad (1)$$

then, one can construct the so-called Szegö function $D_\rho(z)$ associated with the domain E and the weight function $\rho(\xi)$ with the following properties: $D_\rho(z)$ is analytic in Y, $D_\rho(z) \neq 0$ in Y, $D_\rho(\infty) > 0$, $D_\rho(z)$ has limit values on E and

$$|D_\rho(\xi)|^{-p}\, |\Phi'(\xi)| = \rho(\xi),\ \xi \in E \text{ (a.e. on } E.) \quad (2)$$

Where, $D_\rho(\xi) = \lim_{z\to\xi} D_\rho(z)$ (a.e. on E.). Explicitly $D_\rho(z) = D_G(\Phi(z))$, where $D_G(w) = \exp\{ -1/(2p\pi) \int_0^{2\pi}((w+e^{i\theta}) / (w- e^{i\theta})) \log((\rho(\xi)) / |\Phi'(\xi)|)\, |\Phi'(\xi)|\, |d\xi|\}$ $(\xi=\psi(e^{i\theta}))$[6,8,9]. Let f(z) be an analytic function in Y. For p>0, we say that f(z) belongs to $H^p(Y,\rho)$ space if $f((w))/D_\rho(\psi(w))$ is a function from the space $H^p(G)$. For a function F analytic in G, we say that $F\in H^p(G)$ if and only if $F(1/w)\in H^p(U)$, where $w\in U$ and $U=\{w\in C, |w|<1\}$. The space $H^p(U)$ is well known[24-26]. For $p\geq 1$, $H^p(Y,\rho)$ is a Banach space. Each function f(z) from $H^p(Y,\rho)$ has limit values on E and

$$\| f \|^p_{H^p(Y,\rho)} = \int_E |f(\xi)|^p \rho(\xi)|d\xi| = \lim_{R\to 1+} (1/R) \int_{E_R} ((|f(z)|^p)/(|D_\rho(z)|^p))\, |\Phi'(z)\, dz|, \quad (3)$$

Where, $1 < R$, $E_R = \{z\in Y: |(z)| = R\}$.
For $0 < p < \infty$, $H^p(Y,\rho)$ is a metric space with the distance $d(f,g)= \| f-g \|^p_{H^p(Y,\rho)}$.

In what follows we consider $F = E \cup \{z_k\}^\infty_{k=1}$ $(F_\ell = E \cup \{z_k\}^\ell_{k=1})$, $\{z_k\}^\infty_{k=1}$ is an infinite set of points which lay at the exterior of E. Let $\alpha = \beta + \gamma$ ($\alpha_\ell = \beta + \gamma_\ell$) be a finite positive measure on the Borel sets of C, where β and γ (γ_ℓ) are defined as foll ows:

β is a measure concentrated on E and is absolutely continuous with respect to the Lebesgue measure $|d\xi|$ on the arc, i.e.:

$$d\beta(\xi) = \rho(\xi)\, |d\xi|,\ \rho: E\to R_+ \text{ and } \int_E \rho(\xi)\, |d\zeta| < +\infty, \quad (4)$$

and γ (γ_ℓ) is a discrete measure with masses A_k at the points $z_k \in Ext(E)$, k=1, 2,.., i.e.:

$$\gamma = \sum^\infty_{k=1} A_k\delta_{z_k}\ (\gamma_\ell=\sum^\ell_{k=1} A_k\delta_{z_k}),\ A_k>0 \text{ and } \sum^\infty_{k=1} A_k < \infty, \quad (5)$$

Where, δ_{z_k} denotes the (Dirac delta) unit measure supported at the point z_k. By $P_{n,1}$ we denote the set of monic polynomials of degree n. For $p > 0$, we define $m_{n,p}(\alpha,F)$, $m_{n,p}(\alpha_\ell,F_\ell)$, $m_{n,p}(\beta,E)$ and $T_{n,p,\alpha,F}$, $T_{n,p,\alpha\ell,F\ell}$, $T_{n,p,\beta,E} \in P_{n,1}$, as follows:

$$m_{n,p}(\alpha,F) = \|T_{n,p,\alpha,F}\|_{Lp(\alpha,F)} = \inf_{Q_n\in P_{n,1}} \int_E |Q_n(\xi)|^p \rho(\xi)\ |d\xi| + \sum^\infty_{k=1} A_k\ |Q_n(z_k)|^p, \quad (6)$$

$$m_{n,p}(\alpha_\ell,F_\ell) = \| T_{n,p,\alpha\ell,F\ell}\|_{Lp(\alpha\ell,F\ell)} = \inf_{Q_n\in P_{n,1}} \int_E |Q_n(\xi)|^p \rho(\xi)\ |d\xi| + \sum^\ell_{k=1} A_k\ |Q_n(z_k)|^p,$$

$$m_{n,p}(\beta, E) = \| T_{n,p,\beta,E} \|_{Lp(\beta,E)} = \inf_{Q_n\in P_{n,1}} \int_E |Q_n(\xi)|^p \rho(\xi)\ |d\xi| .$$

We suppose that
$$z^n \in Lp(\alpha,F),\ n\in N,\ p>0. \quad (7)$$

Also for $0 < p < \infty$, we define $\mu(\beta)$, $\mu(\alpha)$ and $\mu(\alpha_\ell)$ as the extremal values of the following extremal problems, respectively

$$\mu(\beta) = \inf\{ \| \varphi \|_{H^p(Y,\rho)},\ \varphi\in H^p(Y,\rho),\ \varphi(\infty)=1 \}, \quad (8)$$

$$\mu(\alpha) = \inf\{ \| \varphi \|_{H^p(Y,\rho)},\ \varphi \in H^p(Y,\rho),\ \varphi(\infty)=1 \text{ and } \varphi(z_k)=0,\ k=1,2,\ldots\}, \quad (9)$$

$$\mu(\alpha_\ell) = = \inf\{ \| \varphi \|_{H^p(Y,\rho)},\ \varphi\in H^p(Y,\rho),\ \varphi(\infty)=1 \text{ and } \varphi(z_k)=0,\ k=1,2,..,\ell\}, \quad (10)$$

We denote by φ^* and ψ^* the extremal functions of the problems (8) and (9), respectively and by $B_\infty(z)$ the following Blaschke Product:

$$B_\infty(z) = \prod^\infty_{k=1} ((\Phi(z)- \Phi(z_k)) / (\Phi(z)\, \overline{\Phi(z_k)}-1)) . (\Phi(|(z_k)|^2) / (\Phi(z_k)).$$

The following lemmas summarize some properties of the $H^p(Y,\rho)$ spaces.

Lemma 2.1[19]: If $f(z) \in H^p(Y,\rho)$ and $K\subset Y$, K compact, then there exists a constant $C(K)$ (depending only on K) such that:

$$\sup_K | f(z) | \leq C(K) \| f \|^p_{H^p(Y,\rho)} \quad (11)$$

Lemma 2.2[19]: Let $\{ f_n \}$ be a sequence of functions in $H^p(Y,\rho)$ and
i. $f_n\to f$ uniformly on the compact sets of Y.
ii. $\| f_n \|^p_{Hp(Y,\rho)} \leq M$(const.)
Then
$$F \in H^p(Y,\rho) \text{ and } \| f \|^p_{H^p(Y,\rho)} \leq \liminf_{n\to\infty} \| f_n \|^p_{H^p(Y,\rho)}. \quad (12)$$

Lemma 2.3
i. $B_\infty(z)$ is a bounded analytic function in Y, $B_\infty(\infty) = 1$, $B_\infty(z)$ has a continuous extension to E and $|B_\infty(\xi)| = \prod_{k=1}^\infty |\Phi(z_k)|$.
ii. If $\varphi \in H^p(Y,\rho)$, $\varphi(\infty)=1$ and $\varphi(z_k) = 0$, k=1,2,..., then $f(z) = \varphi(z)/ B_\infty(z) \in H^p(Y,\rho)$ and $f(\infty)=1$.

Proof of Lemma 2.3 : This Lemma was proven in the case p=2[21]. The same proof is also valid in the general case p>0.

Lemma 2.4: The extremal functions φ^* and ψ^* are connected by

$$\Psi^*(z) = \varphi^*(z). \prod^\infty_{k=1}((\Phi(z)- \Phi(z_k)) / (\Phi(z)\, \overline{\Phi(z_k)}-1)). ((|\Phi(z_k)|^2) / (\Phi(z_k))), \quad (13)$$

and

$$\mu(\alpha) = (\prod^{\infty}_{k=1}|\Phi(z_k)|)^p.\mu(\beta). \quad (14)$$

Proof of Lemma 2.4: Consider the function
$f^*(z) = (\Psi^*(z)) / (B_\infty(z)))$,
Lemma 2.3 implies
$f^* \in H^p(Y,\rho)$ and $f^*(\infty) = 1$,
Then $\mu(\beta) \le \| f \|^p_{H^p(Y,\rho)} = (\prod^{\infty}_{k=1} |\Phi(z_k)|)^{-p} \ \| \Psi^* \|^p_{H^p(Y,\rho)}$
$= (\prod^{\infty}_{k=1}|\Phi(z_k)|)^{-p} . \mu(\alpha)$.

Conversely, if we define $g^*(z) = \varphi^*(z) . B_\infty(z)$, then we have

$$g^* \in H^p(Y,\rho), \ g^*(\infty) = 1$$
$$\text{and } g^*(z_k) = 0, \ k=1, 2, ..., \quad (15)$$

(9) and Lemma (2.3) imply $\mu(\alpha) \le \|g^* \|^p_{Hp(Y,\rho)} = (\prod^{\infty}_{k=1} |\Phi(z_k)|)^p \|\varphi^* \|^p_{H^p(Y,\rho)} = (\prod^{\infty}_{k=1} |\Phi(z_k)|)^p \ \mu(\beta)$. So $\mu(\alpha) = (\prod^{\infty}_{k=1} |\Phi(z_k)|)^p \ \mu(\beta)$, and

$$\Psi^*(z) = \varphi^*(z). \prod^{\infty}_{k=1}((\Phi (z)- \Phi(z_k)) / (\Phi(z) \overline{\Phi(z_k)}-1)) . ((|\Phi(z_k)|^2) / (\Phi(z_k))).$$

RESULTS

We can now study the asymptotic behavior of the extremal polynomials $\{ T_{n,p,\alpha,F}(z) \}$. First give some definitions.

For a closed Jordan curve the Faber's polynomials $F_n(z)$ are defined by decomposition

$$\Phi^n(z) = F_n(z) + \lambda_n(z), \text{ with } \lambda_n(z) = O(1/z) \text{ for } z\to\infty.$$

Definition 3.1: A rectifiable Jordan curve E is said to be from the class τ (denoted by $E \in \tau$) if $\lambda_n \to 0$ uniformly on E.

We find[12,27] examples of families of curves belonging to the class τ. If $z:[a,b] \to E$, $z(a) = z(b)$, then a sufficient condition for E to be in the class τ is that $z'(t)$ is in a Lipschitz -class for some exponent.

Definition 3.2: Let $\alpha = \beta + \gamma$, we say that the measure belongs to the class L (denoted by $\alpha\in L$) if the absolute and discrete parts of , satisfy

$$(\sum^{\infty}_{k=1} |\Phi(z_k)-1|) < \infty, \quad (16)$$

and

$$(m_{n,p}(\alpha,F)) / (m_{n,p}(\beta,E)) \le (\prod^{\infty}_{k=1} |\Phi(z_k)|),$$
$$n > N_o, \ p > 0. \quad (17)$$

in addition to conditions (1), (4) and (5). The condition (16) guarantees the convergence of the Blaschke product $B_\infty(z)$ associated to the points $\{z_k\}^{\infty}_{k=1}$.

The condition (17) has been proven in the case $p= 2$ and $E = \Gamma = \{z: |z|=1\}$[23]. One can find a set of points $\{z_k\}^{\infty}_{k=1}$ and $\{A_k\}^{\infty}_{k=1}$ verifying (17).

As an example for $p = 2$ and E the unit circle, we can take $\{z_k\}^{\infty}_{k=1}$ and $\{A_k\}^{\infty}_{k=1}$ as follows:

$$\{z_k \in C : | z_k | = 1 + 1/(k^2) \}^{\infty}_{k=1}, \quad (18)$$

$$\{ A_k = 1/2^k \}^{\infty}_{k=1}, \quad (19)$$

For such sets, Khaldi and Benzine[22] showed that:

$$(m_{n,2}(\alpha_\ell,F_\ell)) / (m_{n,2}(\beta,E)) \le (\prod^{\ell}_{k=1} |z_k|), \ \forall n , \ \forall \ell. \quad (20)$$

and

$$\lim_{\ell\to\infty} m_{n,2}(\alpha_\ell,F_\ell) = m_{n,2}(\alpha,F). \quad (21)$$

(20) and (21) imply

$$(m_{n,2}(\alpha_\ell,F_\ell)) / (m_{n,2}(\beta,E)) \le (\prod^{\ell}_{k=1} |z_k|). \quad (22)$$

We conclude this section by formulating the main result of this study.

Theorem 3.1: If $p>0$, E belongs to the class τ and $\alpha\in L$, then

i. $\lim_{n\to\infty}(m_{n,p}(\alpha,F)) / (C(E)^n) = (\mu(\alpha))^{1/p}$,
ii. $\lim_{n\to\infty}\|(T_{n,p,\alpha,F}(z)/C(E)^n \Phi^n (z))-\Psi^*(z) \|_{H^p(Y,\rho)}=0$,
iii. $T_{n,p,\alpha,F}(z) = C(E)^n\Phi^n(z) \ [\Psi^*(z) \ +\varepsilon_n(z)]$, $\varepsilon_n(z)\to 0$ uniformly on the compact sets of Y.

Proof of Theorem 3.1(i)
$\alpha\in L$, then

$$(m_{n,p}(\alpha,F)) / (C(E)^n) \le$$
$$(m_{n,p}(\beta,E)) / (C(E)^n)).(\prod^{\infty}_{k=1} |\Phi(z_k)|). \quad (23)$$

By using (14) and the fact that

$$\lim_{n\to\infty} (m_{n,p}(\beta,E)) / ((C(E)^n) = (\mu(\beta))^{1/p}, \quad (24)$$

One gets[12,19]

$$\limsup_{n\to\infty} (m_{n,p}(\alpha,F)) / (C(E)^n) \le$$
$$(\mu(\beta))^{1/p} .(\prod^{\infty}_{k=1} |\Phi(z_k)|) = (\mu(\alpha))^{1/p}. \quad (25)$$

It remains to prove that $(\mu(\alpha))^{1/p} \le \liminf_{n\to\infty} (m_{n,p}(\alpha,F)) / (C(E)^n)$. We will present two proofs of this inequality.

First proof: The extremal property of $T_{n,p,\alpha,F}(z)$ and $T_{n,p,\alpha\ell,F\ell}(z)$ imply

$$m_{n,p}(\alpha,F) = \|T_{n,p,\alpha,F}\|_{Lp(\alpha,F)} \ge \|T_{n,p,\alpha,F}\|_{Lp(\alpha\ell,F\ell)} \quad (26)$$
$$\ge \| T_{n,p,\alpha\ell,F\ell}\|_{Lp(\alpha\ell,F\ell)} = m_{n,p}(\alpha_\ell,F_\ell),$$

(26) implies

$$(m_{n,p}(\alpha,F)) / (C(E)^n) \ge (m_{n,p}(\alpha_\ell,F_\ell)) / (C(E)^n)),$$
$$p>0, \ \forall \ell. \quad (27)$$

Using this result and Theorem 2.2[19], we obtain
$\liminf_{n\to\infty} (m_{n,p}(\alpha,F)) / (C(E)^n) \geq (\mu(\alpha_\ell))^{1/p}$,
$p>0, \forall \ell$. (28)

If we take in consideration that
$\mu(\alpha_\ell) = \mu(\beta). (\prod^{\infty}_{k=1} |\Phi(z_k)|)^{p}$[19],
then we get when $\ell \to \infty$

$\liminf_{n\to\infty} (m_{n,p}(\alpha,F)) / (C(E)^n) \geq \mu(\beta)^{1/p}.(\prod^{\infty}_{k=1} |\Phi(z_k)|) = (\mu(\alpha))^{1/p}$. (29)

Another proof : Putting

$$\phi_{n,p}^* = (T_{n,p,\alpha,F}(z) / (C(E)^n \Phi^n, \quad (30)$$

and using (25) we get:

$$\| \phi_{n,p}^* \|^p_{H^p(Y,\rho)} \leq M = \text{const.} \quad (31)$$

Let $M^* = \liminf_{n\to\infty} \| \phi_{n,p}^* \|^p_{H^p(Y,\rho)}$, we have
$M^* = \lim_{n\to\infty, n\in N1\subset N} \| \phi_{n,p}^* \|^p_{H^p(Y,\rho)}$. (32)

This result and Lemma 2.1 imply that $\{\phi_{n,p}^*, n\in N_1\}$ is a normal family in Y. Then, we can find a function $\psi(z)$ that is the uniform limit (on the compact subsets of Y) of some subsequence $\{\phi_{n,p}^*, n\in N_2\}$of $\{\phi_{n,p}^*, n\in N_1\}$. From Lemma 2.2, we get

$\psi \in H^p(Y,\rho)$ and $\|\psi\|^p_{H^p(Y,\rho)}$
$\leq \liminf_{n\to\infty} \| \phi_{n,p}^* \|^p_{H^p(Y,\rho)}$, (33)

On the other hand $\psi(\infty)=1$ and $\psi(z_k)=0$, $k=1,2,...$, (33) implies
$\mu(\alpha) \leq \|\psi\|^p_{H^p(Y,\rho)} \leq \liminf_{n\to\infty} \| \phi_{n,p}^* \|^p_{H^p(Y,\rho)}$
$\leq \liminf_{n\to\infty} ((m_{n,p}(\alpha,F)) / (C(E)^n))^p$. (34)

(34), (25) and (29) imply $(\mu(\alpha))^{1/p} \leq \liminf_{n\to\infty}(m_{n,p}(\alpha,F) / (C(E)^n)) \leq \limsup_{n\to\infty} (m_{n,p}(\alpha,F) / C(E)^n) \leq (\mu(\alpha))^{1/p}$
and (i) follows.

Proof of Theorem 3.1(ii): For the functions $\psi_n = ½ (\phi_{n,p}^* + \psi^*)$, where $\|\psi^*\|^p_{H^p(Y,\rho)} = \mu(\alpha)$, we have $\psi_n(\infty) = 1$ and $\lim_{n\to\infty}, \psi_n(z_k) = 0$, for k =1,2,... As in (i), we get

$$\liminf_{n\to\infty} \| \psi_n \|^p_{H^p(Y,\rho)} \} \geq \mu(\alpha). \quad (35)$$

For $1 \leq p<\infty$, (ii) follows from Clarkson inequality.
For $1\leq p\leq 2$,
$[\int_E |½ (\phi_{n,p}^* + \psi^*)|^p \rho(\xi) |d\xi|]^{1/(p-1)} + [\int_E |½ (\phi_{n,p}^* - \psi^*)|^p \rho(\xi) |d\xi|]^{1/(p-1)} \leq$
$[½ \int_E | \phi_{n,p}^* |^p \rho(\xi) |d\xi| + ½ \int_E | \psi^*|^p \rho(\xi) |d\xi|]^{1/(p-1)}$.
For $2\leq p<\infty$,
$[\int_E |½ (\phi_{n,p}^* + \psi^*)|^p \rho(\xi) |d\xi|]^p + [\int_E |½ (\phi_{n,p}^* - \psi^*)|^p \rho(\xi) |d\xi|]^p \leq$
$½ \int_E | \phi_{n,p}^* |^p \rho(\xi) | d\xi + ½ \int_E | \psi^*|^p \rho(\xi) | d\xi |$

For 0<p<1: In this case one can apply the following extension of the Keldysh Lemma[28]. In our case, the measure is absolutly continuous, so its singular part is equal to zero, then we obtain the following version of Theorem 2[28]. For other details see also [29].

Lemma 3.1[28,29]: Let $\{w_k\}^{\infty}_{k=1}$ be a set of points in G, $\alpha = \beta + \gamma$ such that $\alpha \in L$ and $\{f_n\} \in H^p(G, \rho)$ and $0<p<\infty$. Put

$\bar{f}_n(w) = f_n(w) / \bar{\varphi}^*(w)$, where, $\bar{\varphi}^*(w) = \varphi^*(\psi(w)) = \varphi^*(z) = D_G(w) / D_G(\infty)$,
if

(a) $\lim_{n\to\infty} \bar{f}_n(\infty) = 1$,

(b) $\lim_{n\to\infty} \bar{f}_n(w_k) = 0$, $k = 1,2,...$,
(c) $\sum^{\infty}_{k=1} (| z_k|-1) <+\infty$,
(d) $\liminf_{n\to\infty}\| f_n \|^p_{H^p(G,\rho)} = D_G(\infty) . \prod^{\infty}_{k=1} |w_k|$

Then we have $\lim_{n\to\infty}\| f_n - (\prod^{\infty}_{k=1}(w - w_k/w.\bar{w}_k -1).(|w_k|^2/w_k)).\varphi^*(w) \|_{H^p(G,\rho)} = =0$.

We get (ii) by applying Lemma 3.1 to the sequence

$f_n(w)= \overline{\phi_{n,p}^*}(w)= \phi_{n,p}^*(\psi(w))= \phi_{n,p}^*(z)$,
and the set of points $w_k=\Phi(z_k)$.
We have: $\phi_{n,p}^*(\infty)=1$ and $\varphi^*(\infty)=1$, hence (a) follows. On the other hand (b) is a consequence of the fact that $\varphi^*(z_k) \neq 0$ and $\lim_{n\to\infty} \phi_{n,p}^*(z_k) = 0$, k=1,2,...
(c) is exactly the condition (16) in the case of the circle $\sum^{\infty}_{k=1}(| z_k|-1) <+\infty$, We obtain (d) by considering that
$\mu(\alpha) = (\prod^{\infty}_{k=1} |z_k|)^p . \mu(\beta)$.
and
$\mu(\alpha) = \|\varphi^*\|^p_{H^p(G,\rho)} = \| D_\rho(\infty) / D_\rho \|^p_{H^p(G,\rho)} = | D_\rho(\infty) |^p = | D(\infty) |^p$ and the fact that $\lim_{n\to\infty} \| \phi_{n,p}^* \|^p_{H^p(G,\rho)} =\lim_{n\to\infty} (m_{n,p}(\alpha,F) = (\mu(\alpha))^{1/p}$.

Proof of Theorem 3.1(iii): If we consider the function $\varepsilon_n(z) = (T_{n,p,\alpha,F}(z) / C(E)^n \Phi^n(z)) - \Psi^*(z)$, then (iii) follows from (ii) and Lemma 2.1. This completes the proof of Theorem 3.1.

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