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On Strongly Coupled Linear Elliptic Systems with Application to Otolith Membrane Distortion

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Abstract: Problem Statement: In this research, the author discussed the problems associated with the approximation of the mixed derivative terms appearing in strongly coupled linear elliptic systems by the finite difference method over irregular domains. To avoid the appearance of mixed derivative terms the author introduced a reformulation for the system through introducing a new dependent variable which adds one supplementary (simple) differential equation to the system but does not change its elliptic character. **Approach:** The basic idea in the reformulation is the direct generation of the Laplacian operator which has an efficient finite difference treatment. **Results:** Two finite difference formulae with symmetric appearance approximating the first order derivatives on curved boundaries up to O(h2) are established, that can be considered as a generalization to the well known central formula. Applications to the otolith membrane model have proved the reliability and efficiency of the present treatment in comparison with other methods. **Conclusions/Recommendations:** Although, this treatment has increased the number of algebraic equations approximating the system linearly 3n instead of 2n, the overall accuracy is increased quadratically. The band width of matrix of coefficients of the algebraic system is decreased and there is no need to interpolate along the diagonals due to the absence of mixed derivatives. The treatment is promising and other extensions are mentioned.

Key words: strongly coupled elliptic systems, curved boundaries, irregular domains, finite differences, otolith membrane

INTRODUCTION

In spite of the advances in computer facilities and the extension of their capabilities there is no loss of interest in the efficient algorithms generally and in high-order difference methods. To write an acceptable finite difference model corresponding to a given differential system, some times, it is better to reformulate the system, one way is to write a corresponding first order system and other way is to decompose the differential operator to parts that can be treated efficiently.

The problem of reducing a given system of differential equation to a lower order one or to an equivalent equation with higher order is an old problem. It is completely solved in ordinary differential equations, but in partial differential equations the situation is different, the reduction method is not unique. Laplace's equation and the corresponding Cauchy Riemann equations, the biharmonic equation and the corresponding second order system are good examples,[1,2]. It is well known that the matrix of the algebraic system resulting from approximating the biharmonic equation on a unit square by the finite difference is very ill-posed[2]. Also, there is situations lead to over-determined systems, [2,3].

PDEs with cross-derivative terms arise naturally via the chain rule when an equation without cross-derivative terms on a nonrectangular region is transformed into a rectangular solution domain. The transformation can also introduce cross-derivative terms and convert normal derivative boundary conditions to mixed oblique derivative boundary conditions[4]. Strongly coupled systems of elliptic equations have received considerable attentions in recent years, and various system forms have been proposed in the literature,[5,6,7] and the references therein. We consider in this study a class of linear second order elliptic systems with only two independent variables x- and y-, strongly coupled through the mixed derivative terms $\frac{\partial^2 *}{\partial x \partial y}$ and we introduce a new reformulation, to avoid approximating the mixed derivative terms. This will reduce the band width of the matrix of the algebraic system resulting from the approximation by the finite difference method.

The irregularity of the domains associated with PDE's usually excludes the analytical solutions. Various approaches for the numerical solution of elliptic systems have been considered in the literature[8-11]. Due to the ease of grid generation and the dissipative properties of the Finite Difference Method (FDM), it is the first method one tries to use. However, the FDM usually involves a rectangular grid system, which makes it very difficult to model the detailed topographic features of an irregular domain especially in the existence of mixed derivatives. The Finite Element Method (FEM) can accommodate a more flexible grid work and has been used as an alternative solution scheme for many problems, in some problems the finite element solution is not as stable as the finite difference solution and usually requires the use of nonphysical dissipation(as an example, the elliptic system corresponding to the biharmonic equation)[11]. Furthermore, the generation of a finite element grid with several thousand nodes and with elements of various sizes, shapes and orientations is not a trivial task and you have to use one of the standard software to generate the grid as well as the associated bases. To avoid the difficulties that usually arise from traditional strategies, and also to make use of the efficient treatment in the Laplacian operator, the author tried in this study to generate a Laplacian part plus a simple first order differential operator part. Accordingly, we avoided the use of grid points outside the curved domain and situated along the diagonals increasing the accuracy of the over all finite difference approximations in addition to the structure of the matrix of the corresponding linear algebraic system which will appear through the treatment.

The objective of this study is three fold: in the first we generate a Laplace operator from elliptic differential operators strongly coupled through the mixed derivative term without any transformations, in the second we introduce finite difference formulas and a corresponding difference scheme with high accuracy for problems with curved domains, in the third we applied the treatment to a realistic problem.

Review of elliptic systems: Let us consider the system of partial differential equations:

$$\sum_{j=1}^{n} l_{ij} u_j = F_i \quad \text{in} \quad \Omega, \quad i = 1, \cdots, n$$

Subject to the Dirichlet boundary conditions

$$u_i = 0 \quad \text{on} \quad \partial\Omega, \quad i = 1, \cdots, n$$

Where Ω is a bounded domain in R^2 with boundary $\partial\Omega$, l_{ij} denotes the homogeneous second order differential operator defined by:

$$l_{ij}\, u_j = \frac{\partial}{\partial x}\left(a_{ij}\frac{\partial u_j}{\partial x}\right) + \frac{\partial}{\partial y}\left(b_{ij}\frac{\partial u_j}{\partial x}\right)$$
$$\frac{\partial}{\partial x}\left(b_{ij}\frac{\partial u_j}{\partial y}\right) + \frac{\partial}{\partial y}\left(c_{ij}\frac{\partial u_j}{\partial y}\right)$$

In this study, the author considers the case of determinant constant coefficient systems which can be written in the matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \frac{\partial^2}{\partial x^2}\begin{bmatrix} u \\ v \end{bmatrix} + 2\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \frac{\partial^2}{\partial x \partial y}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \frac{\partial^2}{\partial y^2}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \tag{1}$$

Or in compact form as:

$$A\frac{\partial^2}{\partial x^2}U + 2B\frac{\partial^2}{\partial x \partial y}U + C\frac{\partial^2}{\partial y^2}U = 0 \tag{2}$$

Where u and v are real functions of x, y and A, B and C are 2×2 constant matrices. It is well known that by means of linear transformation of independent variables, linear combination of equations and linear transformation of unknown functions, the system (1) can be reduced to the canonical form which contains at most two independent parameters.
The determinant:

$$F(\xi, \eta) = |\,A\,\xi^2 + 2B\xi\eta + C\eta^2\,| \tag{3}$$

is known as the biquadratic characteristic polynomial of the system (1). This system is classified according to the nature of the roots of the biquadratic characteristic equation

$$F(\tau, 1) = 0, \qquad \tau = \frac{\xi}{\eta} \tag{4}$$

The system (1) is elliptic, when $F(\tau, 1) = 0,$ has a pair of complex roots. It is of the first kind when it has a repeated complex roots, and of the second kind when it has two distinct pairs of complex roots. We will concentrate in this study on systems of the first kind i.e. the biquadratic characteristic equation admits a pair of double complex roots. In this case the system (1) can be written by the use of the above mentioned transformations in a form whose biquadratic characteristic polynomial will take the standard form[5].

$$F(\xi, \eta) = (\xi^2 + \eta^2)^2 \tag{5}$$

Definition 1: The system (1) is said to be reducible if it is equivalent to a system of the form,

$$\begin{aligned}
&\alpha_{11}\frac{\partial^2 u}{\partial x^2} + 2\beta_{12}\frac{\partial^2 u}{\partial x \partial y} + \gamma_{11}\frac{\partial^2 u}{\partial y^2} = 0 \\
&\alpha_{22}\frac{\partial^2 v}{\partial x^2} + 2\beta_{22}\frac{\partial^2 v}{\partial x \partial y} + \gamma_{22}\frac{\partial^2 v}{\partial y^2} = \\
&-(\alpha_{21}\frac{\partial^2 u}{\partial x^2} + 2\beta_{21}\frac{\partial^2 u}{\partial x \partial y} + \gamma_{21}\frac{\partial^2 u}{\partial y^2})
\end{aligned} \tag{6}$$

i.e., the study of reducible systems is equivalent to the study of two simple equations successively, otherwise it is irreducible, and thus irreducible systems of the form (1) are strongly coupled through the term with mixed derivatives, which will be our interest in this study. Thus, irreducible systems can be written in the form:

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix} u \\ v \end{bmatrix} + 2\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\frac{\partial^2}{\partial x \partial y} \\
&\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}\frac{\partial^2}{\partial y^2}\begin{bmatrix} u \\ v \end{bmatrix} = 0
\end{aligned} \tag{7}$$

With

$$\begin{aligned}
&b_2 \neq 0 \quad \text{and} \quad b_3 \neq 0 \\
&b_1 + b_4 = 0 \\
&\lambda + \mu + 4b_1 b_4 - 4b_2 b_3 = 2 \\
&\mu\, b_1 + \lambda\, b_4 = 0 \\
&\mu\, \lambda = 1
\end{aligned} \tag{8}$$

In the finite difference approximation for the system (7) each equation will contain at least 9 grid unknowns and for domains with curved boundaries the corner grid points due to the mixed derivatives will be outside the domain when using the finite differences near the boundary. Accordingly, reducing the accuracy of the over all system.

MATRIALS AND METHODS

The use of a local five-point approximation scheme making use of only neighboring nodes of a square grid and avoiding mesh points situated in the diagonal of the associated linear difference operator which is known to be optimal for Laplace operators[12], will be my objective in this study. Accordingly, the bandwidth of the coefficient matrix of the associated linear system is reduced and the accuracy is increased when treating problems with curved boundaries. Despite of the irregular shape of the otolith membrane, the boundary conditions can be realized in a way preserving the consistency of the finite difference equations with the system of differential equations and giving a global approximation error up to the fourth order.

Formulation of the problem: One of the difficulties in handling problems with finite differences is the curved domains especially with the existence of mixed derivatives. In the approximation of the mixed derivative terms we use data along the diagonals of the grid system. So the author introduced a reformulation for the system (7) to eliminate the mixed derivative term. The author distinguishes between two cases:

Case 1: $\lambda \neq \mu$, we must have $b_1 = b_4 = 0$ accordingly the first canonical form can be written as:

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix} u \\ v \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} \\
&\frac{\partial^2}{\partial x \partial y}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}\frac{\partial^2}{\partial y^2}\begin{bmatrix} u \\ v \end{bmatrix} = 0
\end{aligned} \tag{9}$$

with

$$\lambda\,\mu = 1, \qquad \text{and} \qquad \lambda + \mu - 4b = 2 \tag{10}$$

Where, $b = b_2 b_3$

$$\begin{aligned}
&u_{xx} + \lambda\, u_{yy} + 2\, v_{xy} = 0 \\
&v_{xx} + \frac{1}{\lambda} v_{yy} + 2\, b\, u_{xy} = 0
\end{aligned} \tag{11}$$

We define

$$w = (\lambda - 1)\, u_y + 2\, v_x \tag{12}$$

Then with the help of (10), this system, can be written in the form

$$\begin{aligned}
&u_{xx} + u_{yy} + w_y = 0 \\
&v_{xx} + v_{yy} + \frac{\lambda - 1}{2} w_x = 0 \\
&(\lambda - 1)\, u_y + 2\, v_x - w = 0
\end{aligned} \tag{13}$$

Case 2: $\lambda = \mu \neq 1$ (if $\lambda = \mu = 1$ the system will be reducible) and we must have $\lambda = \mu = -1$ and accordingly $b_4 = -b_1$, and the system becomes

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix}u\\v\end{bmatrix}+2\begin{bmatrix}b_1 & b_2\\b_3 & -b_1\end{bmatrix}\frac{\partial^2}{\partial x\partial y}\begin{bmatrix}u\\v\end{bmatrix}+\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\frac{\partial^2}{\partial y^2}\begin{bmatrix}u\\v\end{bmatrix}=0 \quad (14)$$

Which can be further transformed to give the second canonical form

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix}u\\v\end{bmatrix}+2\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\frac{\partial^2}{\partial x\partial y}\begin{bmatrix}u\\v\end{bmatrix}+\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\frac{\partial^2}{\partial y^2}\begin{bmatrix}u\\v\end{bmatrix}=0 \quad (15)$$

With

$$\lambda\mu = 1, \quad \text{and} \quad \lambda+\mu-4b = 2. \quad (16)$$

$$\begin{aligned} u_{xx} - u_{yy} + 2v_{xy} &= 0\\ v_{xx} - v_{yy} - 2b\,u_{xy} &= 0 \end{aligned} \quad (17)$$

We define

$$w = 2u_y - 2v_x \quad (18)$$

Then the above system can be written in the form

$$\begin{aligned} u_{xx} + u_{yy} - w_y &= 0\\ v_{xx} + v_{yy} + w_x &= 0\\ u_y - v_x - \frac{1}{2}w &= 0 \end{aligned} \quad (19)$$

In systems (13) and (19) we don't have mixed derivative terms, but we have an extra first order differential equation as we will see in the application below. The number of the algebraic equations approximating the system of PDE is increased linearly "3n instead of 2n" the accuracy for curved domains is increased quadratically and the band width of the sparse matrix of coefficients is decreased "at most 7 instead of at least 9". We will consider the relation between the band width and the dimension of the coefficient matrix for like systems in a subsequent work.

The mathematical model of the otolith membrane: The otolith membrane in man is apart of the vestibular organ which controls the subjective sensation of equilibrium, spatial orientation and motion. It also influences different vegetative functions such as blood pressure and coagulation time and it stabilizes the eyes in space during a movement of the head[13].

The otolith membrane is a thin layer of a gelatinous substance which covers a plaque of hair cells called the macula utricle. Adherent to the elastic membrane are a large number of calcite cristals known as the otoconia. They have a much higher specific mass than the otolith membrane and the macula. When the head is in an upright position, the macula is nearly horizontal. Any movement of the head causes accelerations and therefore exerts forces of different strength on macula and otoconia. This induces shearing forces and produces a distortion of the otolith membrane. By evaluating this distortion, some indication of compensating movements of the head for restoring the balance and staiblizing the eyes can be obtained.

The investigations might be interesting in connetion with space exploration research, when the behavior of the sensory apparatus has to be studied under different gravitational conditions.

Following Hudetz[14] and Youssef *et al.*[15], the otolith membrane is considered as a flat, thin, ovoid structure which is fixed at the boundary. Movements are restricted to the plane of the membrane which is likewise the plane of the macula. No displacements perpendicular to this plane are possible. It is supposed to be situated in the x-y plane of a 3-dimensional Cartesian coordinate system if the head is in an upright position (parallel to the z-axis).

The irregular boundary Γ of the membrane Ω can be approximated by the curves C_1 , C_2 which are connected at the points $A = (-0.064, 0.096)$ and $B = (-0.064, -0.096)$, Twizell[16], Twizell and Curran[17], Castillo *et al.*[18,19], Youssef *et al.* [15] as shown in Fig. 1.

$$C_1: \quad \frac{x^2}{(0.096+\varepsilon)^2} + \frac{y^2}{(0.128)^2} = 1 \quad (20)$$

Where

$$\begin{aligned} \varepsilon &= 0.000758905 \quad &&\text{for} \quad -0.064 \le x \le 0.07\\ \varepsilon &= 0.0 \quad &&\text{for} \quad x \ge 0.07 \end{aligned}$$

and

$$C_2: \quad x = -0.084 - 0.5\,y + 2.1701\,y^2 + 54.2535\,y^3, \quad |y| \le 0.096 \quad (21)$$

This definition shifts some annoying boundary discontinuity observed in the above literature from nodes A (= no. 4) and B (= no. 43) to points which are not mesh points.

It is generally accepted[14-19] that the displacements u and v in the x- and y- directions of all points of the membrane are governed within the domain of the membrane by the linear elliptic system of partial differential equations.

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial y} = -\frac{1-\nu^2}{E}F^{(x)} \tag{22}$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial y} = -\frac{1-\nu^2}{E}F^{(y)} \tag{23}$$

together with the Dirichlet boundary conditions

$$u(x,y) = v(x,y) = 0 \quad \text{for all} \quad (x,y) \in \Gamma; \quad \Gamma = C_1 \cup C_2 \tag{24}$$

$F^{(x)}$ and $F^{(y)}$ are the x and y components of the gravitational force which is considered as being the only acting force (static conditions) and is given by

$$\begin{aligned} F^{(x)} = \rho\, g\, [&\cos\psi\,(\sin\alpha \sin\gamma + \cos\alpha \sin\beta \cos\gamma) \\ &+ \sin\psi(\sin\alpha \cos\gamma - \cos\alpha \sin\beta \sin\gamma)] \end{aligned} \tag{25}$$

$$\begin{aligned} F^{(y)} = \rho\, g\, [&-\sin\psi\,(\sin\alpha \sin\gamma + \cos\alpha \sin\beta \cos\gamma) \\ &+ \cos\psi\,(\sin\alpha \cos\gamma - \cos\alpha \sin\beta \sin\gamma)] \end{aligned} \tag{26}$$

where ψ is the azimuth angle and α, β, γ are the angles of rotation of the membrane about the *x*-, *y*-, *z*- axes.

The model constants are the Poisson ratio ν which is chosen as 0.5 (For gelatinous substances it lies in the range $0.3 \le \nu \le 0.8$), the Young's modulus $E = 200000\ \text{dynes}\,\text{cm}^{-3}$ the density $\rho = 0.903\ \text{gm}\,\text{cm}^{-3}$ the earth gravity constant $g = 981\ \text{cm}\,\text{sec}^{-2}$ The maximum length of the membrane is 0.256 cm and the maximum width 0.192 cm. It is easily proved that the system is elliptic in the sense of Petrowski for all such ν,[2,5,7].

Since the irregularity of the boundary excludes any analytical solution of the system it must be solved by numerical methods. This will be done by finite differnce techniques after superimposing a square grid, with grid constant $h = 0.032$ cm, on a rectangular region including the membrane domain Ω (Fig. 1).

Exactly the same grid is used by Twizel[16] and castillo *et al.*[18], Youssef *et al.*[15]. The mesh points lie at the intersections of the lines $x = 0.032m,\ y = 0.032n\ (m, n = 0, \pm 1, \pm 2, \cdots)$. There are thirty interior grid points in Ω and twenty boundary points on Γ as shown in Fig. 1.

Finite difference scheme for the Otolith membrane model: The appearance of the mixed derivatives in (22) and (23) introduces mesh points situated in the diagonal of the associated linear difference operator and the matrix of the corresponding finite difference system, which in general is sparse, will get four supplementary non-zero entries in every row. This in general reduces the average rate of convergence if iterative methods are chosen for the solution of the linear finite difference system. Define an auxiliary function $w = \frac{1+\nu}{2}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right]$ the mixed second order derivatives in (22), (23) can be eliminated and one gets the following system of partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial w}{\partial y} = -\frac{1-\nu^2}{E}F^{(x)} \tag{27}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial x} = -\frac{1-\nu^2}{E}F^{(y)} \tag{28}$$

$$-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{2}{1+\nu}w = 0 \tag{29}$$

With

$$u(x,y) = v(x,y) = 0 \quad \text{for all} \quad (x,y) \in \Gamma \tag{30}$$

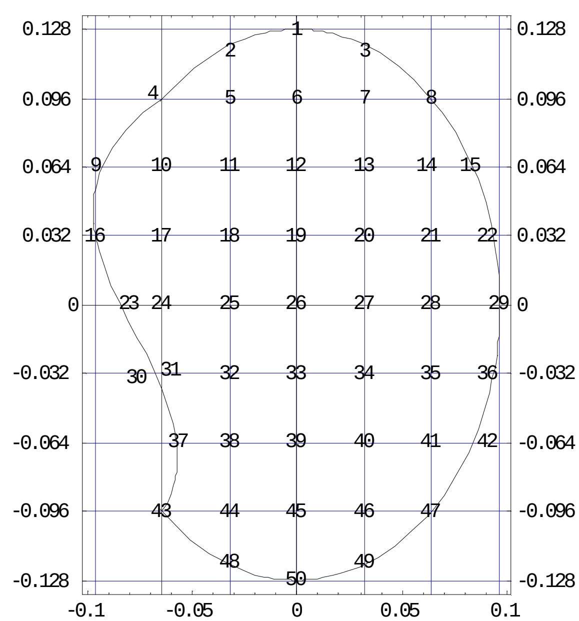

Fig. 1: The grid imposed to the otolith membrane domain

This system is much simpler than the original partial differential system (22, 23). It is still linear elliptic of the second order, but the principal part of the equations (27), (28) is now defined only by the classical Laplacian which is easily handled in discretization. The supplementary differential equation (29) is of the first order and does not represent any difficulty in describing its finite difference approximation of accuracy up to the second order. It is interesting to note that, for this system and in accordance with equation (10), we have $\nu = 0.5$, $b_2 = 0.375$, $b_3 = 1.5$ $\lambda = 0.25$, $\mu = 4$ and $b = 0.5625$

Now, denoting the values of a function *s* at node $(0.032m, 0.032n)$ by $s_{m,n}$ and assuming sufficient differentiability of *s*, finite difference approximations for the derivatives can be obtained by the well known Taylor development:

$$\left.\frac{\partial s}{\partial x}\right|_{m,n} = \frac{s_{m+1,n} - s_{m-1,n}}{2h} + O(h^2) \tag{31}$$

$$\left.\frac{\partial s}{\partial y}\right|_{m,n} = \frac{s_{m,n+1} - s_{m,n-1}}{2h} + O(h^2) \tag{32}$$

$$\left.\frac{\partial^2 s}{\partial x^2}\right|_{m,n} = \frac{s_{m+1,n} - 2s_{m,n} + s_{m-1,n}}{h^2} + O(h^2) \tag{33}$$

$$\left.\frac{\partial^2 s}{\partial y^2}\right|_{m,n} = \frac{s_{m,n+1} - 2s_{m,n} + s_{m,n-1}}{h^2} + O(h^2) \tag{34}$$

If the y-derivative is needed at a point $(mh, (n+\theta)h)$, $0 < \theta < 1$, which is not necessarily a mesh point (see Fig. 2), it can be approximated by

$$\left.\frac{\partial u}{\partial y}\right|_{m,n+\theta} = \frac{-(2+\frac{2}{\theta})u_{m,n} + \frac{2+4\theta}{\theta+\theta^2}u_{m,n+\theta} + \frac{2\theta}{1+\theta}u_{m,n-1}}{2h} + O(h^2) \qquad 0 < \theta \le 1 \tag{35}$$

If the *x*-derivative is needed at a point $(mh, (n+\theta)h)$, $0 \le \theta < 1$, it can be approximated by

$$\left.\frac{\partial u}{\partial x}\right|_{m,n+\theta} = \frac{(1+\theta)u_{m+1,n} - (1+\theta)u_{m-1,n} - \theta u_{m+1,n-1} + \theta u_{m-1,n-1}}{2h} + O(h^2) \qquad 0 \le \theta < 1 \tag{36}$$

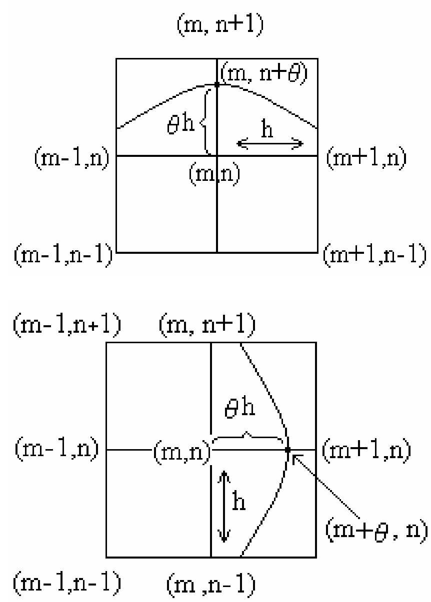

Fig. 2: boundary points and not grid points

The same formulae can be used analogously to approximate the derivatives $\left.\frac{\partial u}{\partial x}\right|_{m+\theta,n}$ and $\left.\frac{\partial u}{\partial y}\right|_{m+\theta,n}$. Note that formulas (36) generalize formula (31) for $\theta \neq 0$. The finite difference representation of the system of equations (27), (28) and (29) is

$$u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} - \frac{h}{2}(w_{m,n+1} - w_{m,n-1}) = h^2\left(-\frac{1-\nu^2}{E}F^{(x)}\right)_{m,n} \tag{37}$$

$$v_{m-1,n} + v_{m+1,n} + v_{m,n-1} + v_{m,n+1} - 4v_{m,n} + \frac{h}{2}(w_{m+1,n} - w_{m-1,n}) = h^2\left(-\frac{1-\nu^2}{E}F^{(y)}\right)_{m,n} \tag{38}$$

$$u_{m,n+1} - u_{m,n-1} - v_{m+1,n} + v_{m-1,n} - \frac{4h}{1+\nu}w_{m,n} = 0 \tag{39}$$

It is easily seen that these equations are consistent with the system of partial differential equations and the accuracy of the scheme is $O(h^2)$. The above equations are applied to the thirty nodes in Ω and a system of 90 equations is obtained. This system is banded (sparse) with band width smaller by at least 2 than the standard one used by Twizel[16].

When the above system is applied to nodes adjacent to the boundary sometimes function values at mesh points outside of Ω must be expressed by values at internal and boundary nodes. This is easily achieved by using interpolating polynomials of the second degree along grid lines in both directions x and y and conserving second order accuracy of the approximation.

Equation (29) is used with formulae (35) and (36) to approximate the boundary values of w.

The linear finite difference system is quickly solved by successive overrelaxation.

RESULTS AND DISCUSSION

The objective of this study is three fold: in the first, the author generates a Laplace operator from elliptic differential operators strongly coupled through the mixed derivative term without any transformations, in the second the author introduces finite difference formulas and a corresponding difference scheme with high accuracy for problems with curved domains, in the third the author applied the treatment to a real problem. The objective has been successively treated.

In this study the author introduced a simple trick based on the simple form of the biquadrate characteristic form to generate the well known Laplace differential operator from strongly coupled linear elliptic system in which the coupling is due to the mixed derivative terms. The mixed derivatives can not be eliminated by the three well known transformations for like systems, Hua *et al.*[5]. Also, the author introduced two simple finite difference formulae (35) and (36) with symmetric appearance approximating the first order derivatives on curved boundaries up to $O(h^2)$ that can be considered as a generalization to the well known central formulae. Accordingly, we used the five-point difference operator approximating the Laplacian on a square grid which is known to be optimal, and of the second order, Birkhoff[12]. We applied our treatment to a realistic problem, which appear when modeling the distortion of the otolith membrane under gravity forces. After elimination of the mixed derivatives no diagonal nodes of the central difference operator occur. Since the values at mesh points neighboring the boundary (from inside and outside) are also approximated with $O(h^2)$. The total finite difference equation system has an order of consistency up to the second order.

The absolute value of the displacement $d_{m,n}$ at the point $(0.032\,\text{m}, 0.032\,\text{n})$ is $d_{m,n} = \sqrt{(u_{m,n})^2 + (v_{m,n})^2}$, its slope with regard to the positive x-axis is $(v_{m,n} / u_{m,n})$. A displacement $d_{m,n}$ is considered as positive if the calculated value of v is positive. Otherwise it is negative.

Three sets of numerical experiments were performed.

In the first one the membrane was rotated in steps of $\alpha = \pi/12$ through 2π about the *x*-axis starting from its normal equilibrium position, β and γ were maintained at 0. Physically the head was turned through 360° in steps of 15° by first raising the right ear (Fig. 3).

In the second set the membrane was rotated in steps of $\beta = \pi/12$ through 2π about the y-axis starting from its normal equilibrium position, α and γ were maintained at 0. Physically the head was turned from its normal upright position through 360° in steps of 15° by first raising the nose (Fig. 4).

In the third set the acceleration due to gravity was allowed to vary according to Table 1, thus modeling gravity conditions on the surfaces of Earth, Moon, Mars, Venus and Jupiter (Fig. 5).

In contrast to Twizell's results[16], no perturbation of the sinusoidal behavior of the displacement curves has been observed in the neighborhood of α = 45° (see Fig. 3).

Table 1: Acceleration due to gravity on the surface of Earth, Moon, Mars, Venus and Jupiter

Surface	G (cm sec^{-2})
Earth	981
Moon	162
Mars	373
Venus	873
Jupiter	2551

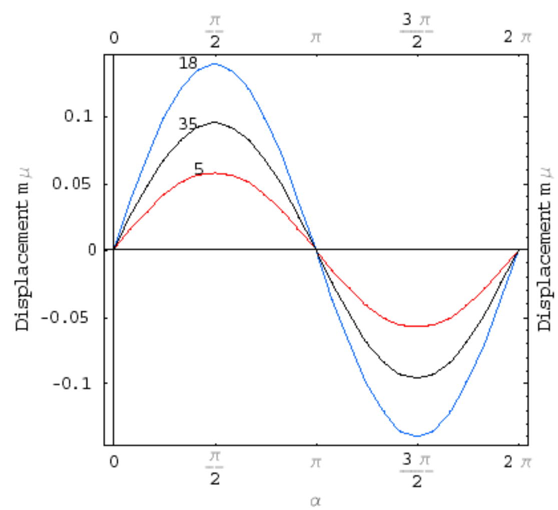

Fig. 3: Displacements $d_{m,n}$ for nodes 5, 18, 35 as a function of α on the surface of the Earth

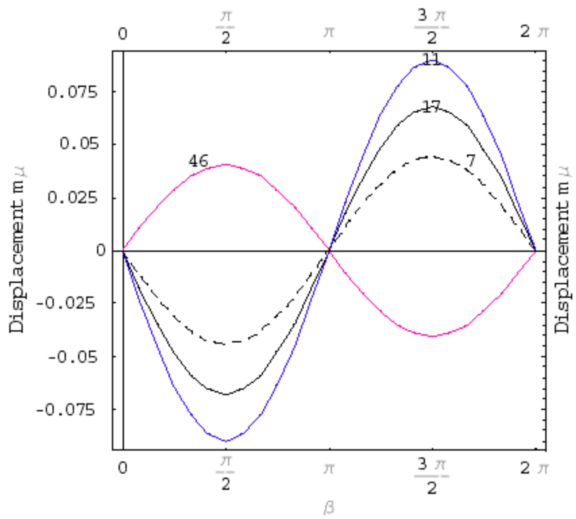

Fig. 4: Displacements $d_{m,n}$ for nodes 7, 11, 17, 46 as a function of β on the surface of the Earth

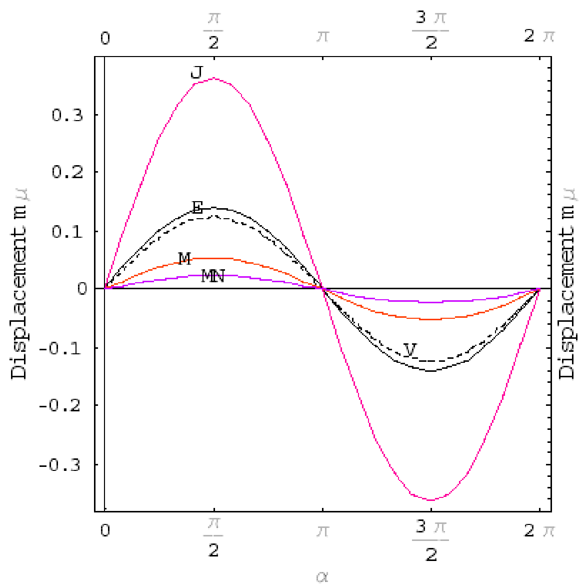

Fig. 5: Displacements $d_{m,n}$ for the node 18 as a function of α on the surfaces of Jupiter (J), Earth (E), Venus (V), Mars (M), and Moon (MN)

This expected result might be explained by the fact that the discontinuity in the definition of the boundary curves has been transferred in the present study from nodes A and B to places which are no more mesh points.

The results presented in (Figs. 3, 4 and 5) are in full agreement with those obtained by Castillo *et al.*[19] and Youssef *et al.*[15] in applying standard finite element, isoperimetric finite element method and variational techniques.

CONCLUSION

The objective of the study has completely achieved. The finite difference formulas introduced can be used in other problems with curved boundaries. The efficient methods used in treating the Laplace operators can be used directly to problems with mixed derivatives.

In comparison with the results of Twizell[16] which is problem specific and only obtains a first order accuracy due to linear Lagrange interpolation for mesh points adjacent to the boundary this treatment is more accurate. Moreover the traditional computational work is straight forward; the banded matrix of the finite difference system is sparse and well adapted for iterative solution methods and has band width at most 7 instead of at least 9. Although, the number of equations in the algebraic system is increased linearly (3n) instead of (2n) the accuracy is increased quadratically.

In comparison with the results of Youssef *et al.*[15] , Twizel and Curran[17], Castillo *et al.*[18,19] with the finite element, isoperimetric finite element, boundary element and boundary fitted coordinates, the computational work as well as the theoretical treatment introduced in this study is more straightforward easy, convenient and at the same time gives the same results. The numerical results illustrate the high efficiency and reliability of the treatment. This approach can be applied with slight modifications to problems which are not elliptic or strongly coupled because the mathematical work in this studyis more simple than any other transformations.

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