Journal of Mathematics and Statistics 5 (3):178-182, 2009
ISSN 1549-3644

Hypercyclic Functions for Backward and Bilateral Shift Operators

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Abstract: Problem statement: Giving conditions for bilateral forward and unilateral backward shift operators over the weighted space of p-summable formal series to be hypercyclic. This provides a generalization to the case of Hilbert space. **Approach:** We used hypercyclicity criterion and some preliminary concepts for formal Laurent series and formal power series. Moreover we got benefits of some duality properties of above mentioned spaces. **Results:** We obtained necessary and sufficient conditions for bilateral forward and unilateral backward shift operators to be hypercyclic. **Conclusion:** The bilateral forward shift operator was hypercyclic on the space of all formal Laurent series and the unilateral backward shift operator was hypercyclic on the space of all formal power series under certain conditions.

Key words: Bilateral shift operators, weighted shift operators, hypercyclic functions, hypercyclic operators

INTRODUCTION

A vector x in a Banach space X is called hypercyclic for a bounded operator T if the orbit $\{T^n x : n \geq 0\}$ is dense in X. The first examples of hypercyclic operators appeared in the space of entire functions defined over the complex plane, endowed with the compact-open topology. In 1929 Birkhoff[1] essentially showed the hypercyclicity of the translation operators $T_a f(z) = f(z+a), a \neq 0$, while MacLane[2] proved the hypercyclicity of the differentiation operator. The notion of hypercyclicity on Banach spaces started in 1969 with Rolewicz[3], who showed that any scalar multiple λB of the unilateral backward shift B is hypercyclic on $l^p(1 \leq p<\infty)$ and c_0, whenever $|\lambda| > 1$. Kitai in his thesis with title invariant closed sets for linear operators, university of Toronto, determined conditions under which a linear operator is hypercyclic. This result, commonly referred to as the hypercyclicity criterion, was never published and a few years later it was rediscovered in a broader form by Gethner and Shapiro[5]. During the last year's hypercyclicity criterion on Banach or Frechet spaces has attracted many mathematicians working in linear functional analysis and very important contributions to the topic have been made[5-13].

We use the hypercyclicity criterion of[10] G. Godefroy and J. H. Shapiro (1991), to show that the bilateral forward shift operators on the space of all formal Laurent series and the unilateral backward shift operators on the space of all formal power series are hypercyclic. Our results in the case *p* = 2 are compatible to that given by H. Salas (1995) on the space $l^2(Z)$ and generalize those given by[12].

MATERIALS AND METHODS

Let $\{\beta(k)\}$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space of all functions $f(z) = \sum_k a_k \frac{z^k}{\beta(k)}$ such that:

$$\|f\| = \left(\sum_k |a_k|^p\right)^{\frac{1}{p}} < \infty \tag{1}$$

if k ranges only the nonnegative integers, these are formal power series, otherwise they are formal Laurent

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series, where $\frac{z^k}{\beta(k)}$ is an orthonormal basis. We shall denote these spaces by:

H^p_β : Power series case

E^p_β : Laurent series case

Definition: The operator M_z on E^p_β defined by $M_z f(z) = zf(z)$ is called the bilateral forward shift; furthermore, the inverse of M_z which is the bilateral backward shift is the operator B_z defined on E^p_β by $B_z f(z) = \frac{f(z)}{z}$. The right inverse of M_z is the unilateral backward shift operator S_z defined on H^p_β by,

$$S_z f(z) = \frac{f(z) - f(0)}{z}$$

Noting that the shift operator defined over the space E^p_β by $R_z f(z) = \sum_k a_k \frac{z^{k+1}}{\beta(k+1)}$ is a shift operator, for which $\|R_z\| = 1$.

Since $\beta(k) > 0, \forall k \in Z$, then by taking $w_k = \frac{\beta(k+1)}{\beta(k)}$ we get $\beta(n) = \prod_{k=0}^{n-1} w_k$ and $\beta(n) = \prod_{k=0}^{n-1} (w_{-k})^{-1} \forall n \in N$ and the operator T defined by $Te_k = w_k e_{k+1}$ is an injective bilateral forward weighted shift operator on $l^p(Z) = \left\{x : x = \sum_{k\in Z} x_k e_k,\ \|x\|^p = \sum_{k\in Z} |x_k|^p < \infty\right\}$ with weight sequence $\{w_k\}_{k=-\infty}^{\infty}, w_k \neq 0$. On the other hand the operator V defined by $Ve_k = \frac{1}{w_{k-1}} e_{k-1}$ is an injective backward weighted shift operator on $l^p(Z)$ with weight sequence $\{w_k\}_{k=-\infty}^{\infty}, w_k \neq 0$.

Proposition 1: The bilateral forward shift operator M_z on E^p_β is unitarily equivalent to the injective bilateral forward weighted shift operator T mentioned above. Conversely, every injective bilateral forward weighted shift operator T is unitarily equivalent to M_z acting on E^p_β.

Proof: We define the unitary operator U: $l^P(Z) \to E^P_\beta$ by:

$$Ue_k := \frac{z^k}{\beta(k)}$$

Hence:

$$U^* \frac{z^k}{\beta(k)} = e_k$$

Then:

$$U^* M_z U e_k = w_k e_{k+1}$$

So, M_z is unitarily equivalent to an injective bilateral forward weighted shift operator (with the weight sequence $\{w_k\}$).

Conversely, since T is injective bilateral forward weighted shift with weight w_n, then $\beta(n) \neq 0$ for all n. Without loss of generality[4], we may assume that the shifts have positive weights $\beta(n)$. Noting that:

$$TU^* \frac{z^k}{\beta(k)} = w_k e_{k+1}$$

We get:

$$UTU^* = M_z$$

Proposition 2: The bilateral backward shift operator B_z on E^p_β is unitarily equivalent to the injective weighted shift linear operator V mentioned above (with the weight sequence $\{(w_{k-1})^{-1}\}$). Conversely, every injective weighted shift operator is unitarily equivalent to B_z acting on E^p_β.

The proof of this result is similar to that of proposition 1.

Proposition 3: The powers M^n_z of the operator M_z are bounded on E^p_β with $\|M^n_z\| = \sup_k \frac{\beta(k+n)}{\beta(k)}$ provided that the supremum in the right hand side exists.

Proof: For $f \in E^p_\beta$, it is seen that $M^n_z f(z) = \sum_{k=-\infty}^{\infty} a_k \frac{z^{k+n}}{\beta(k)}$ and hence:

$$\|M^n_z f\| = \left(\sum_{k=-\infty}^{\infty} |a_k|^P \left(\frac{\beta(k+n)}{\beta(k)}\right)^P\right)^{1/P} \leq \sup_k \frac{\beta(k+n)}{\beta(k)} \|f\|$$

Thus:

$$\|M_z^n\| \le \sup_k \frac{\beta(k+n)}{\beta(k)}$$

On the other hand $\left\|\frac{z^{k+n}}{\beta(k)}\right\| \le \|M_z^n\|$ and so $\frac{\beta(k+n)}{\beta(k)} \le \|M_z^n\|$; hence $\sup_k \frac{\beta(k+n)}{\beta(k)} \le \|M_z^n\|$ and the result holds.

In the same manner we can prove that $\|B_z^n\| = \sup_k \frac{\beta(k-n)}{\beta(k)}$ and $\|S_z^n\| = \sup_k \frac{\beta(k)}{\beta(k+n)}$ provided that the right hand side exists in each case.

Proposition 4: A sufficient condition for the series $\sum_{k=-\infty}^{\infty} a_k \frac{z^k}{\beta(k)}$ to be convergent is:

$$\lim_{k\to\infty} \sup_k \frac{1}{\left(\beta(-k)\right)^{1/k}} < |z| < \lim_{k\to\infty} \inf_k \left(\beta(k)\right)^{1/k} \quad (2)$$

Proof: Let $f(z) = \sum_{k=-\infty}^{\infty} a_k \frac{z^k}{\beta(k)} \in E_\beta^p$. Since condition (2) is sufficient for the convergence of the series $\sum_k \frac{|z|^{kq}}{\beta^q(k)}$ for every z, then by using Hölder inequality we get:

$$|f(z)| = \left|\sum_k a_k \frac{z^k}{\beta(k)}\right| \le \left(\sum_k |a_k|^p\right)^{1/P} \left(\sum_k \frac{|z|^{kq}}{\beta^q(k)}\right)^{1/q}$$

This is the required result.

One can easily see that the dual of E_β^p is E_γ^q, where $\frac{1}{p}+\frac{1}{q}=1$ and $\gamma = \{\gamma(k)\} = \left\{(\beta(k))^{-1}\right\}$. In fact, for any functional U on E_β^p, there exist an element $g(z) = \sum_k b_k \frac{z^k}{\gamma(k)} \in E_\gamma^q$ such that

$U(f(z)) = \langle f(z), g(z)\rangle = \sum_k a_k b_k$ for any $f(z) = \sum_k a_k \frac{z^k}{\beta(k)}$, where $\langle f(z), g(z)\rangle$ denote the value of the functional $g(z) \in E_\gamma^q$ on the element $f(z) \in E_\beta^p$. According to this notation we get:

$$\left\langle \frac{z^k}{\beta(k)}, \frac{z^m}{\gamma(m)} \right\rangle = \delta_{k,m}$$

RESULTS

Now we give necessary and sufficient conditions for bilateral forward shift operator on the space E_β^p is hypercyclic.

Proposition 5: The bilateral forward shift operator M_z on E_β^p is hypercyclic if and only if for every ε > 0 and $q \in N$, there exists sufficiently large $n_i \in N$, such that for every $|k| \le q$:

$$\frac{\beta(k+n_i)}{\beta(k)} < \varepsilon \quad \text{and} \quad \frac{\beta(k)}{\beta(k-n_i)} > \frac{1}{\varepsilon} \quad (3)$$

Proof: Assume that M_z is hypercyclic. Since the set of hypercyclic functions for M_z is dense, then given ε > 0 and $q \in N$, let $0 < \delta < 1$ so that $0 < \delta/(1-\delta) < \varepsilon$ and there is a hypercyclic function f(z) for M_z such that:

$$\left\| f(z) - \sum_{|k|\le q} \frac{z^k}{\beta(k)} \right\| < \delta$$

Using the functionals $\frac{z^m}{\gamma(m)}$ with norm one in the dual space of E_β^p such that:

$$\left|\left\langle f(z) - \sum_{|k|\le q} \frac{z^k}{\beta(k)}, \frac{z^k}{\gamma(k)} \right\rangle\right| \le \left\| f(z) - \sum_{|k|\le q} \frac{z^k}{\beta(k)} \right\| < \delta$$

We get:

$$|a_k| < \delta, \qquad |k| > q \quad (4)$$

$$|a_k| > 1-\delta, \qquad |k| \le q \quad (5)$$

Since f(z) is hypercyclic, we can choose $n_i > 2q$ such that:

$$\left\| M_z^{n_i} f(z) - \sum_{|k|\le q} \frac{z^k}{\beta(k)} \right\| < \delta \quad (6)$$

Let $|k| \le q$ be fixed. Since $n_i > 2q$ inequality (6) implies that:

$$\left\|\left\langle M_z^{n_i} f(z) - \sum_{|k|\le q} \frac{z^k}{\beta(k)}, \frac{z^{k+n_i}}{\gamma(k+n_i)} \right\rangle\right\| = \frac{\beta(k+n_i)}{\beta(k)} |a_k| < \delta$$

From (4) and (5) we get:

$$\frac{\beta(k+n_i)}{\beta(k)} < \frac{\delta}{1-\delta} < \varepsilon$$

On the other hand, inequality (6) implies that:

$$\left\| \left\langle M_z^{n_i} f(z) - \sum_{|k|\le q} \frac{z^k}{\beta(k)}, \frac{z^k}{\gamma(k)} \right\rangle \right\| = \left| a_{k-n_i} \frac{\beta(k)}{\beta(k-n_i)} - 1 \right| < \delta$$

Therefore:

$$\frac{\beta(k)}{\beta(k-n_i)} > \frac{1-\delta}{a_{k-n_i}} > \frac{1-\delta}{\delta} > \frac{1}{\varepsilon}$$

Conversely, let $Y_0 := \text{span}\left\{\frac{z^k}{\beta(k)}, k \in Z\right\}$ be a dense subset of E_β^p and let $B_z : Y_0 \to Y_0$ be the linear mapping defined by:

$$B_z\left(\frac{z^k}{\beta(k)}\right) := \frac{z^{k-1}}{\beta(k)} \tag{7}$$

Notice that $B_z M_z = M_z B_z = Id_{Y_0}$. It is easy to see that $M_z^{n_i} \to 0$ point-wisely on Y_0. On the other hand by using (3) we get:

$$\left\| B_z^{n_i} \sum_k a_k \left(\frac{z^k}{\beta(k)}\right) \right\|^p = \sum_k |a_k|^p \left(\frac{\beta(k-n_i)}{\beta(k)}\right)^p \xrightarrow[i\to\infty]{} 0$$

Thus the hypercyclicity criterion is satisfied and the proof is complete.

DISCUSSION

Results obtained from proposition 5 can be applied in the next proposition but for the unilateral backward shift operator on the space H_β^p.

Proposition 6: The unilateral backward inverse shift operator S_z on H_β^p is hypercyclic if and only if for every $\varepsilon > 0$ and $q \in N$, there exists sufficiently large $n_i \in N$, such that for every $k \in N$:

$$\frac{\beta(k+n_i)}{\beta(k)} < \varepsilon \tag{8}$$

Proof: Assume that S_z is hypercyclic. Since the set of hypercyclic functions for S_z is dense, then given $\varepsilon > 0$ and $q \in N$, let $0 < \delta < 1$ so that $0 < \delta/(1-\delta) < \varepsilon$ and there is a hypercyclic function f(z) for S_z such that:

$$\left\| f(z) - \sum_{k=0}^{q} \frac{z^k}{\beta(k)} \right\| < \delta$$

Choosing suitable functionals in the dual space of E_β^p we get:

$$|a_k| < \delta, \qquad k > q \tag{9}$$

$$|a_k| > 1-\delta, \qquad 0 \le k \le q \tag{10}$$

Since $f(z)$ is hypercyclic, we can choose $n_i > q$ such that:

$$\left\| S_z^{n_i} f(z) - \sum_{k=0}^{q} \frac{z^k}{\beta(k)} \right\| < \delta \tag{11}$$

Let $0 \le k \le q$ be fixed. From (9), (10) and (11) we get:

$$1-\delta < \left| a_{k+n_i} \right| \frac{\beta(k)}{\beta(k+n_i)} < \delta \frac{\beta(k)}{\beta(k+n_i)}$$

Therefore:

$$\frac{\beta(k+n_i)}{\beta(k)} < \frac{\delta}{1-\delta} < \varepsilon$$

Conversely let $X_0 :=$ span $\left\{\frac{z}{\beta(1)}, \frac{z^2}{\beta(2)}, ...\right\}$ be a dense subset of H_β^p and let $A_z : X_0 \to X_0$ be the linear mapping defined by:

$$A_z\left(\frac{z^k}{\beta(k)}\right) := \frac{z^{k+1}}{\beta(k)} \tag{12}$$

Notice that $(S_z A_z) f(z) = f(z)$ and $S_z^{n_i} \to 0$ point-wisely on X_0. On the other hand by using (8) we get:

$$\left\| A_z^{n_i} \sum_{k=0}^{\infty} a_k \left(\frac{z^k}{\beta(k)}\right) \right\|^p = \sum_{k=0}^{\infty} |a_k|^p \left(\frac{\beta(k+n_i)}{\beta(k)}\right)^p \xrightarrow[i\to\infty]{} 0$$

Thus the hypercyclicity criterion is satisfied and the proof is complete.

CONCLUSION

The bilateral forward shift operator is hypercyclic on the space of all formal Laurent series and the unilateral backward shift operator is hypercyclic on the space of all formal power series under certain conditions. Propositions 5 and 6 in the case of $p = 2$ are compatible with that given in[7] over the space $l^2(Z)$ and in[10] over the space $l^2(N)$. Noting that, some examples of hypercyclic bounded linear operators have applications in physics and quantum radiation field theory[10,12].

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