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Edge Double-Critical Graphs

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Abstract: Problem statement: The vertex double-critical conjecture that the only vertex double-critical graph is the complete graph has remained unresolved for over forty years. The edge analogue of this conjecture has been proved. **Approach:** It was observed that if the chromatic number decreases by two upon the removal of a 2-matching, then the 2-matching comprises four vertices which determine an induced subgraph isomorphic to the complete graph on four vertices. This observation was generalized to t-matchings. **Results:** In this note, it has been shown that the only edge double-critical graph is the complete graph. **Conclusion/Recommendations:** An alternate proof that the only edge double-critical graph is the complete graph has been obtained. Moreover, the result has been obtained independently.

Key words: Chromatic number, critical clique, k-matching

INTRODUCTION

The graphs considered in this study are finite, undirected and simple. For a given graph G, the vertex and edge sets of G are denoted by V(G) and E(G), respectively. The order of G, denoted by $n = |V(G)|$, is the cardinality of V(G). An r-clique is a complete subgraph of order r and is denoted by K_r. A subset M of E(G) is said to be independent whenever no two edges in M share a common vertex. In case $|M| = k$, the set M is called a k-matching. For a subset X of V(G), the subgraph of G induced by X is denoted by G[X]. All vertex colorings are proper, i.e., a partition of V(G) into independent subsets of V(G) called color classes. Lastly, $\chi(G)$ denotes the chromatic number of G and is the minimum cardinality of a partition of V(G) determined by a proper vertex coloring of G.

A graph G is said to be vertex double-critical provided $\chi(G\text{-}v) = \chi(G)\text{-}2$ for every adjacent pair of vertices u, v. This definition arises out of its relation to the Erdos-Lovasz Tihany Conjecture. A special case of this conjecture is that the only vertex double-critical graph is the complete graph; it is often referred to as the Erdos-Lovasz double-critical conjecture. A brief discussion of the Erdos-Lovasz Tihany Conjecture and related results for quasi-line graphs are given in (Balogh *et al*., 2009). Stiebitz (1987) has shown that K_5 is the only 5-chromatic vertex double-critical graph. To date, the Erdos-Lovasz double-critical conjecture remains open for k-chromatic graphs with $k \geq 6$. In Theorem 6 of (Kawarabayashi *et al*., 2010), the edge analogue of the Erdos-Lovasz double-critical conjecture is proved. This note offers an independent proof of the edge analogue of the Erdos-Lovasz double-critical conjecture. Relations to FTTMs and the inertia tensor of a tetrahedron as defined in (Ahmad *et al*., 2010; Tonon, 2005), respectively are also being investigated.

Edge double-critical graphs: It is now shown that K_n is the only edge double-critical graph. First, some notational conventions and a required definition are given. Let $M_t = \{e_1, e_2, \ldots, e_t\}$ be a set of t edges in E(G) and set $e_i = u_i v_i$ for $i = 1, 2, \ldots, t$. Next, define $M_t^* = \{u_1, v_1\} \cup \{u_2, v_2\} \cup \cdots \cup \{u_t, v_t\}$. Clearly, M_t is a t-matching when $|M_t| = 2t$.

Definition 1: Let G be a graph which contains 2-matchings. Then G is called edge double-critical whenever $\chi(G\text{-}M_2) = \chi(G)\text{-}2$ for every 2-matching M_2.

Necessarily, an edge double-critical graph is connected. An important observation is given in the following lemma.

Lemma 1: Let $M_2 = \{e_1, e_2\}$ be a 2-matching such that $\chi(G\text{-}M_2) = \chi(G)\text{-}2$. Then $G[M_2^*] \cong K_4$.

Proof: Set $k = \chi(G)$ and let $e_i = u_i v_i$ for $i = 1, 2$. Consider any (k-2)-coloring of $G\text{-}M_2$, the colors being from among $\{c_1, c_2, \ldots, c_{k-2}\}$. Then u_1 and v_1 must be colored the same since otherwise there would exist a (k-2)-coloring of $G\text{-}e_2$. A similar argument shows that u_2 and v_2 must be colored the same, necessarily using a different color from that used for u_1 and v_1. Next, observe that $u_1u_2 \in E(G\text{-}M_2)$. Else, both u_1 and u_2 could be recolored using color c_{k-1}. But this would allow e_1 and e_2 to be added back to $G\text{-}M_2$ resulting in a coloring of G using fewer than k colors. A similar argument shows that u_1v_2, v_1u_2, $v_1v_2 \in E(G\text{-}M_2)$. Consequently, $G[M_2^*] \cong K_4$.

Theorem 1: Let $t \geq 1$. If $\chi(G\text{-}M_t) = \chi(G)\text{-}t$, then M_t is a t-matching of G. Moreover, $G[M_t^*] \cong K_{2t}$.

Proof: Let $k = \chi(G)$. The result is trivial for $t = 1$. Let $t \geq 2$ and consider a subset M_t of E(G) such that $\chi(G\text{-}M_t) = k\text{-}t$. Because $|M_t| = t$, it follows that M_t is a t-matching as incident edges can decrease the chromatic number of a graph by at most one upon their removal. Observe now that for all pairs i, j with $i \neq j$, $\chi(G\text{-}e_i\text{-}e_j) = k\text{-}2$. By setting $M_2(i, j) = \{e_i, e_j\}$ and applying Lemma 1, $G[M_2^*(i, j)] \cong K_4$. Hence, $G[M_t^*] \cong K_{2t}$.

Proposition 1: Every t-matching in K_{2t} is critical.

Proof: The proof is by induction on t. For t = 1, the result is trivial. Since $\chi(K_4\text{-}M) = \chi(C_4) = 2$ for every 2-matching M of K_4, Proposition 1 holds for t = 2. Now, inductively assume that Proposition 1 holds for t = 1, 2, … , t′ -1. Let $M_{t'}$ be any t′-matching in $K_{2t'}$. Notice that $K_{2t'}$ can be written as $K_{2t'} = K_2 + K_{2(t'-1)}$. Moreover, it can be assumed, without loss of generality, that the single edge in the K_2 term is in the t′-matching $M_{t'}$. Consequently, $M_{t'}$ can be written as $M_{t'} = M_1 \cup M_{t'\text{-}1}$, where M_1 is the 1-matching in the K_2 term and $M_{t'\text{-}1}$ is a (t′-1)-matching in the $K_{2(t'\text{-}1)}$ term. By the inductive hypothesis, $\chi(K_{2(t'\text{-}1)}\text{-}M_{t'\text{-}1}) = t'\text{-}1$. Therefore:

$$K_{2t'}\text{-}M_{t'} = (K_2+K_{2(t'-1)})\text{-}(M_1 \cup M_{t'-1}) = (K_2\text{-}M_2) + (K_{2(t'-1)}\text{-}M_{t'-1}) = E_2 + (K_{2(t'-1)}\text{-}M_{t'-1})$$

Hence, $\chi(K_{2t'}\text{-}M_{t'}) = 1 + (t'\text{-}1) = t'$.

Corollary 1: Every matching in K_n, $n \geq 2$, is critical.

Lemma 1 and Corollary 1 together set the stage for the main result of this note.

Theorem 2: G is edge double-critical if and only if $G \cong K_n$, provided $n \geq 4$.

Proof: If $G \cong K_n$, where $n \geq 4$, then by Corollary 1, every 2-matching in K_n is critical. Thus, G is edge double-critical. Conversely, let G by a connected, edge double-critical graph. Take any $u, v \in V(G)$ and suppose to the contrary that $uv \notin E(G)$. Then $N(u) = N(v) = \{w_{u,v}\}$ for some vertex $w_{u,v} \in V(G)$. Otherwise, because G is connected, it would follow that $u, v \in M_2^*$ for some 2-matching M_2. Since G is edge double-critical, $G[M_2^*] \cong K_4$ by Lemma 1. This implies that $uv \in E(G)$, contrary to our supposition. Next, observe that $N(z) = \{w_{u,v}\}$ for every vertex $z \neq w_{u,v}$. Else, by using exactly the same argument as above, we would be forced to conclude that $z \in N(u) = \{w_{u,v}\}$, which is clearly not possible by the choice of z. The above argument leads to the conclusion that G is a star. But such a graph is known not to be edge double-critical because of the absence of 2-matchings in any star. Hence, $uv \in E(G)$ so that $G \cong K_n$.

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