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MOMENTS OF NONIDENTICAL ORDER STATISTICS FROM BURR XII DISTRIBUTION WITH GAMMA AND NORMAL OUTLIERS

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ABSTRACT

There are some distributions with no simple closed form for distribution functions such as the Normal and Gamma distributions. This will be the problem if we want to find moments of nonidentical order statistics in the presence of Gamma and Normal outliers observations. We used the idea of approximating Normal and Gamma distributions with Burr type XII distribution. We get single moments for order statistics from sample of independent nonidentically distributed Burr XII random variables that contains p-outlier from Normal or Gamma distributions. Approximating these distributions with Burr XII distribution and then we compared the results by previous method.

Keywords: Approximation, Nonidentical Order Statistics, Burr XII Distribution**,** Normal Distribution, PERMINANTS, Gamma Distribution

1. INTRODUCTION

There are no simple closed form exists for the normal distribution function and the gamma distribution function so that approximations to G (x) must be used to find moments of the rth identical order statistic. An approach of obtaining close approximation to the normal distribution was presented by Burr (1942). Burr (1967; 1973), Burr recalculated the values of shape parameters for Burr XII distribution. These values give a closer approximation to the normal distribution. Another approach based on approximating gamma distribution with the Burr family distribution has been presented by Tadikamalla and Ramberg (1975) and Wheeler (1975), who approximate Gamma with two parameters with Burr with two parameters. Tadikamalla (1977) used the generalized four parameter B-distribution to approximate gamma distribution and put a table for selected integral values of shape parameter and other parameters which Burr XII distribution function approximate to the exact Gamma distribution function. In this chapter we used the idea of approximating Normal and Gamma distributions with Burr XII to get single moments for order statistics from sample of independent nonidentically distributed Burr XII random variables that contains p-outlier from normal or Gamma distributions. approximating these distributions with Burr XII distribution and then we compared the results by those using the Barakat and Abdelkader (2004).

Barnett and Lewis (1994) have defined an outlier in a set of data to be “an observation” or subset of observations” which appears to be inconsistent with the remainder of the set of data”. They also describe several models for outlier; see (Moshref and Sultan, 2007). Density functions and joint density functions of order statistics arising from a sample of a single outlier have been given by Shu (1978) and Hartley and David (1978). One may also refer to Vaughan and Venables (1972) for more general expressions of distributions of order statistics using permanent expressions. Arnold and Balakrishnan (1989) have obtained the density function of Xr:n when the sample of size n contains unidentified

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single outlier. They also obtained the joint density function of $X_{r:n}$ and $X_{s:n}$,$1\le r<s\le n$. Balakrishnan and Balasubramanian (1995) has derived some recurrence relations satisfied by the single and product moments of order statistics from the right truncated exponential distribution. Also he has deduced the recurrence relations for the multiple outlier models (with slippage of observations), see also Balakrishnan (1994). Childs *et al*. (2001) have derived some recurrence relations for the single and product moments of order statistics from n independent and non-identically distributed Lomax and the right-truncated Lomax random variables.

We assume $X_1, X_2,\ldots,X_p$ are independent with probability density function f (x) while X_n-p+1,…,X_n are independent were arise from some modified version of f (x) which call g(x) in which the shape parameters have been shifted in value. Finally, some special cases are deduced.

The probability density function of the rth order statistics $X_{r:n}$, under the multiple outlier model can be written as, see Childs (1996) Equation 1:

$$f_{r:n}[p](x)=\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1 f(x)\{F(x)\}^s\{G(x)\}^{r-s-1}\times\{1-F(x)\}^{n-p-s-1}\{1-G(x)\}^{p-r+s+1}+\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2 g(x)\{F(x)\}^s\{G(x)\}^{r-s-1}\times\{1-F(x)\}^{n-p-s}\{1-G(x)\}^{p-r+s},\quad 1\le r\le n,\ p=0,1,2,\cdots n,\ -\infty<x<\infty \tag{1}$$

Where Equation 2:

$$C_1=\frac{(n-p)!p!}{s!(r-s-1)!(n-p-s-1)!(p-r+s+1)}\qquad C_2=\frac{(n-p)!p!}{s!(r-s-1)!(n-p-s)!(p-r+s)} \tag{2}$$

Setting p = 1 in (1) we obtain the corresponding pdf's in case of the single outlier given by Shu (1978).

In this study, we consider the case when the variable $X_1, X_2,\ldots, X_{n-p}$ are independent observations from Burr XII with four parameters distribution with density Equation 3:

$$f(x)=\frac{\rho c}{b}(1+\left(\frac{x-a}{b}\right)^c)^{-\rho-1}\left(\frac{x-a}{b}\right)^{c-1},\quad a\le x\le\infty,\rho>0,c>0,b>0 \tag{3}$$

and $X_{n-p+1},\ldots, X_n$ arise from the same distribution with density Equation 4:

$$g(x)=\frac{\tau c}{b}(1+\left(\frac{x-a}{b}\right)^c)^{-\tau-1}\left(\frac{x-a}{b}\right)^{c-1},\quad a\le x\le\infty,\tau>0,c>0,b>0 \tag{4}$$

The corresponding cumulative distribution functions F (x) and G(x) are given as Equation 5 and 6:

$$F(x)=1-(1+\left(\frac{x-a}{b}\right)^c)^{-\rho},\quad x\ge a,\rho>0,c>0,b>0 \tag{5}$$

$$G(x)=1-(1+\left(\frac{x-a}{b}\right)^c)^{-\tau},\quad a£x,\tau>0,c>0,b>0 \tag{6}$$

The relation between f (x) and F (x) is given by Equation 7:

$$f(x)=\frac{\rho c}{b}\left(\frac{x-a}{b}\right)^{c-1}[1-F(x)]-\left(\frac{x-a}{b}\right)^c f(x) \tag{7}$$

Similarly, the relation between g(x) and G(x) is Equation 8:

$$g(x)=\frac{\tau c}{b}\left(\frac{x-a}{b}\right)^{c-1}[1-G(x)]-\left(\frac{x-a}{b}\right)^c g(x) \tag{8}$$

In the following section, we use (3) and (4) to derive the single and product moments of order statistics from Burr XII distribution under the multiple outlier models. This situation is known as a multiple outlier model with slippage of p observations; Barnett and Lewise (1994). This specific multiple outlier model was introduced by Launer and Bills (1979).

1.1. Single Moments

We derive the k_{th} moment of the r_{th} order statistic under multiple outlier models (with a slippage of p observations). Let μ (k) [p]; (1≤r≤n) denote the k_{th} single moments of order statistics in the presence of p-outlier observations from BurrXII distribution The following theorem gives an explicit form of μ(k) [p].

Theorem 1

For 1≤r≤ n, p = 0, 1,…, n and k = 0,1,… the single moments $\mu^{(k)}_{r:n}[p]$ is given by Equation 9:

$$\begin{aligned}
\therefore\ \mu^{(k)}_{r:n}[p] &= \rho b^k \sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} \\
& C_1 \sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \\
& \times \sum_{i=0}^{k}\left(\frac{a}{b}\right)^i\binom{k}{i}\beta\left(\phi-\frac{k}{c}+\frac{i}{c}-1,\frac{k}{c}-\frac{i}{c}+1\right) \\
& + \tau b^k \sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} \\
& C_2 \sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \\
& \times \sum_{i=0}^{k}\left(\frac{a}{b}\right)^i\binom{k}{i}\beta\left(\phi-\frac{k}{c}+\frac{i}{c}-1,\frac{k}{c}-\frac{i}{c}+1\right)
\end{aligned} \tag{9}$$

where, $0^0 = 1$, $b \neq 0$ Equation 10:

$$\begin{aligned}
&\phi = \rho(n-p-s+1)+\tau(p-r+s+l+m)+1, \\
&C_1 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s-1)!(p-r+s+1)} \\
&C_2 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s)!(p-r+s)}
\end{aligned} \tag{10}$$

Proof

Starting from (1), we have Equation 11:

$$\begin{aligned}
\mu^{(k)}_{r:n}[p] &= \int_0^{\infty} x^k f_{r:n}[p](x)dx \\
&= \sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1 \int_0^{\infty} x^k f(x)\{F(x)\}^s\{G(x)\}^{r-s-1} \\
&\{1-F(x)\}^{n-p-s-1}\{1-G(x)\}^{p-r+s+1}dx \\
&+ \sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2 \int_0^{\infty} x^k g(x)\{F(x)\}^s\{G(x)\}^{r-s-1} \\
&\{1-F(x)\}^{n-p-s}\{1-G(x)\}^{p-r+s}dx
\end{aligned} \tag{11}$$

Using the relations Equation 12:

$$\begin{aligned}
&[1-F(x)]^{-1} = \frac{\frac{\rho c}{b}\left(\frac{x-a}{b}\right)^{c-1}}{f(x)\left(1+\left(\frac{x-a}{b}\right)^c\right)} \\
&g(x) = \frac{\frac{\tau c}{b}[1-G(x)]\left(\frac{x-a}{b}\right)^{c-1}}{\left(1+\left(\frac{x-a}{b}\right)^c\right)}
\end{aligned} \tag{12}$$

We get Equation 13 and 14:

$$\begin{aligned}
\mu^{(k)}_{r:n}[p] &= \sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1\int_a^{\infty} \\
& x^k f(x)\{F(x)\}^s\{G(x)\}^{r-s-1} \\
& \{1-F(x)\}^{n-p-s} \\
& \times \frac{\frac{\rho c}{b}\left(\frac{x-a}{b}\right)^{c-1}}{f(x)\left(1+\left(\frac{x-a}{b}\right)^c\right)} \\
& \{1-G(x)\}^{p-r+s+1}dx \\
& + \sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2\int_a^{\infty} \\
& x^k \frac{\frac{\tau c}{b}[1-G(x)]\left(\frac{x-a}{b}\right)^{c-1}}{\left(1+\left(\frac{x-a}{b}\right)^c\right)} \\
& \{F(x)\}^s\{G(x)\}^{r-s-1} \\
& \times\{1-F(x)\}^{n-p-s}\{1-G(x)\}^{p-r+s}dx
\end{aligned} \tag{13}$$

$$\begin{aligned}
\mu^{(k)}_{r:n}[p] &= \frac{\rho c}{b^c}\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} \\
& C_1\int_a^{\infty} x^k(x-a)^{c-1}\left(1+\left(\frac{x-a}{b}\right)^c\right)^{-1} \\
& \{F(x)\}^s\{G(x)\}^{r-s-1}\times \\
& \{1-F(x)\}^{n-p-s}\{1-G(x)\}^{p-r+s+1}dx \\
& + \frac{\tau c}{b^c}\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} \\
& C_2\int_a^{\infty} x^k(x-a)^{c-1}\left(1+\left(\frac{x-a}{b}\right)^c\right)^{-1} \\
& \{F(x)\}^s\{G(x)\}^{r-s-1}\times\{1-F(x)\}^{n-p-s} \\
& \{1-G(x)\}^{p-r+s+1}dx
\end{aligned} \tag{14}$$

Now by writing:

F (x) = 1- (1-F (x)) and G (x) = 1-(1-G (x)) in (14) and expand we get Equation 15:

$$\mu_{r:n}^{(k)}[p]=\frac{\rho c}{b^{c}}\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1\int_a^\infty x^k (x-a)^{c-1}(1+\left(\frac{x-a}{b}\right)^c)^{-1} \sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \times\{1-F(x)\}^{n-p-s+1}\{1-G(x)\}^{p-r+s+1+m}dx + \frac{\tau c}{b^{c}}\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2\int_a^\infty x^k (x-a)^{c-1}(1+\left(\frac{x-a}{b}\right)^c)^{-1}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \times\{1-F(x)\}^{n-p-s+1}\{1-G(x)\}^{p-r+s+1+m}dx \quad (15)$$

We know that Equation 16-18:

$$1-F(x)=(1+\left(\frac{x-a}{b}\right)^c)^{-\rho} \quad 1-G(x)=(1+\left(\frac{x-a}{b}\right)^c)^{-\tau}$$

$$\mu_{r:n}^{(k)}[p]=\frac{\rho c}{b^{c}}\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1\int_a^\infty x^k (x-a)^{c-1}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \times(1+\left(\frac{x-a}{b}\right)^c)^{-[\rho(n-p-s+1)]+\tau(p-r+s+1+m)+1]}dx + \frac{\tau c}{b^{c}}\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2\int_a^\infty x^k (x-a)^{c-1}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \times(1+\left(\frac{x-a}{b}\right)^c)^{-[\rho(n-p-s+1)]+\tau(p-r+s+1+m)+1]}dx \quad (16)$$

Le:

$$\frac{1}{y}=1+\left(\frac{x-a}{b}\right)^c\Rightarrow\left(\frac{x-a}{b}\right)^c=\frac{1-y}{y}\Rightarrow$$

$$\left(\frac{x-a}{b}\right)=\left(\frac{1-y}{y}\right)^{\frac{1}{c}}\Rightarrow x-a=b\left(\frac{1-y}{y}\right)^{\frac{1}{c}}$$

$$\therefore x=b\left(\frac{1-y}{y}\right)^{\frac{1}{c}}+a$$

$$dx=\frac{b}{c}\left(\frac{1-y}{y}\right)^{\frac{1}{c}-1}\left(-\frac{1}{y^2}\right)dy$$

$$\text{at } x=a\Rightarrow y=1,\ \text{at } x=\infty\Rightarrow y=0$$

$$\mu_{r:n}^{(k)}[p]=\rho b^k\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m}\times\int_0^1\left[b\left(\frac{1-y}{y}\right)^{\frac{1}{c}}+a\right]^k \times y^{[\rho(n-p-s+1)+\tau(p-r+s+1+m)+1]-2}dy + \tau b^k\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2\sum_{l=2}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m}\times\int_0^1\left[b\left(\frac{1-y}{y}\right)^{\frac{1}{c}}+a\right]^k \times y^{[\rho(n-p-s+1)]+\tau(p-r+s+1+m)+1]-2}dy \quad (17)$$

Let:

$$\left[b\left(\frac{1-y}{y}\right)^{\frac{1}{c}}+a\right]^k=\sum_{i=0}^{k}\binom{k}{i}a^i\left(\frac{1-y}{y}\right)^{\frac{k-i}{c}}b^{k=i}$$

$$\therefore\ \mu_{r:n}^{(k)}[p]=\rho b^k\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)}$$

$$C_1\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m}$$
$$\times\sum_{i=1}^{k}\left(\frac{a}{b}\right)^i\binom{k}{i}\int_0^1 (1-y)^{\frac{k-i}{c}}$$
$$[\rho(n-p-s+l)|+$$
$$\tau(p-r+s+1+m)+1]-\frac{k-i}{c}-2$$
$$\times y\ dy$$
$$+\tau b^k\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)}$$
$$C_2\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m}$$
$$\times\sum_{i=0}^{k}\left(\frac{a}{b}\right)^i\binom{k}{i}\int_0^1 (1-y)^{\frac{k-i}{c}}$$
$$[\rho(n-p-s+l)|+ \tag{18}$$
$$y^{\tau(p-r+s+1+m)+1]-\frac{k-i}{c}-2}\,dy$$

Let Φ = ρ(n-p-s+l)+τ(p-r+s+1+m)+1 Equation 19:

$$\int_0^1 (1-y)^{\frac{k-i}{c}}\,y^{\Phi-\frac{k-i}{c}-2}\,dy$$
$$=\beta\left(\Phi-\frac{k-i}{c}-2,\frac{k-i}{c}+1\right)$$
$$\therefore \mu_{r:n}^{(k)}[p]=\rho b^k\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)}$$
$$C_1\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m}$$
$$\times\sum_{i=0}^{k}\left(\frac{a}{b}\right)^i\binom{k}{i}\beta(\phi-\frac{k}{c}+\frac{i}{c}-1,\frac{k}{c}-\frac{i}{c}+1)$$
$$+\tau b^k\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)}C_2\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1} \tag{19}$$
$$\binom{r-s-1}{m}(-1)^{l+m}$$
$$\times\sum_{i=0}^{k}\left(\frac{a}{b}\right)^i\binom{k}{i}\beta(\phi-\frac{k}{c}+\frac{i}{c}-1,\frac{k}{c}-\frac{i}{c}+1)$$

Remark

If we put a = 0, b = 1 in (9) and consider $0^0 = 1$ we get moments of order statistics from Burr XII distribution with two parameters in the presence of outlier observations as Equation 20:

$$\therefore \mu_{r:n}^{(k)}[p]=\rho\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)}$$
$$C_1\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}$$
$$(-1)^{l+m}\ \beta(\phi-\frac{k}{c}-1,\frac{k}{c}+1)$$
$$+\tau\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)}C_2\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}$$
$$\binom{r-s-1}{m}(-1)^{l+m}\ \beta(\phi-\frac{k}{c}-1,\frac{k}{c}+1) \tag{20}$$

4 Special Cases

We deduce some special cases from the single moments given in (9) and (20) as follows:

- Setting p = 0, we get the single moments of order statistics when $x_1, x_2, \ldots, x_n$ have Burr distribution as Equation 21 and 22:

$$\mu_{r:n}^{(k)}[0]=\frac{\rho b^k n!}{(r-1)!(n-r)!}\sum_{l=0}^{r-1}\binom{r-1}{l}$$
$$(-1)^l\sum_{i=0}^{k}\left(\frac{a}{b}\right)^i\binom{k}{i} \tag{21}$$
$$\beta(\phi-\frac{k}{c}+\frac{i}{c}-1,\frac{k}{c}-\frac{i}{c}+1)$$
$$\Phi=\rho(n-r+1+l)+1$$

$$\mu_{r:n}^{(k)}[0]=\frac{\rho n!}{(r-1)!(n-r)!}\sum_{l=0}^{r-1}\binom{r-1}{l}$$
$$(-1)^l\ \beta(\phi-\frac{k}{c}+\frac{i}{c}-1,\frac{k}{c}-\frac{i}{c}+1) \tag{22}$$
$$\Phi=\rho(n-r+1+l)+1$$

- If we put p = n, we have the same relations above but with parameter τ
- If we put p = 1, we have the relations for single outlier

Example (1)

Let X_1, X_2~Barr XII with c = 1.39533, ρ = 3, a = 0.13445, b = 15.67968 and X_3~Gamma(β = 1, α = 2). Find $\mu_{1:3}$.

Solution

A single outlier in this sample that X_3~Gamma(β = 1, α = 2).

Table 1. Results of Tadikamalla (1977) for the parameters ρ, a, b, c, μ, σ when α is integer and β = 1

α	k	c	a	b	μ	σ
2.0	17.97966	1.39533	0.13445	15.67968	2.001763475	1.415469000
*3.0	11.87409	1.66677	0.34929	12.56702	3.001531924	1.733370325
4.0	9.81594	1.87242	0.62394	12.34255	4.002643944	2.001365465
*5.0	8.78044	2.03679	0.94586	12.74163	5.003001406	2.237449340
6.0	8.15747	2.17284	1.30639	13.34974	6.003306453	2.450905466
8.0	7.44596	2.38816	2.12026	14.75852	8.003844356	2.829869466
10.0	-7.05312	2.55363	3.03161	16.21132	10.004322450	3.163752577
15	6.57114	2.84543	5.61319	19.67139	15.011104120	3.874534021
25	-6.22985	3.18708	11.58794	25.68629	24.999992660	4.999998172
50	6.02306	3.58956	28.87270	37.49438	49.999986320	7.071055695
100	5.96191	3.91545	67.62502	54.86029	100.000013700	9.999999494
∞ Normal	6.15784	4.87370	-3.97998	6.17322	-1.21E-07000	0.999999908

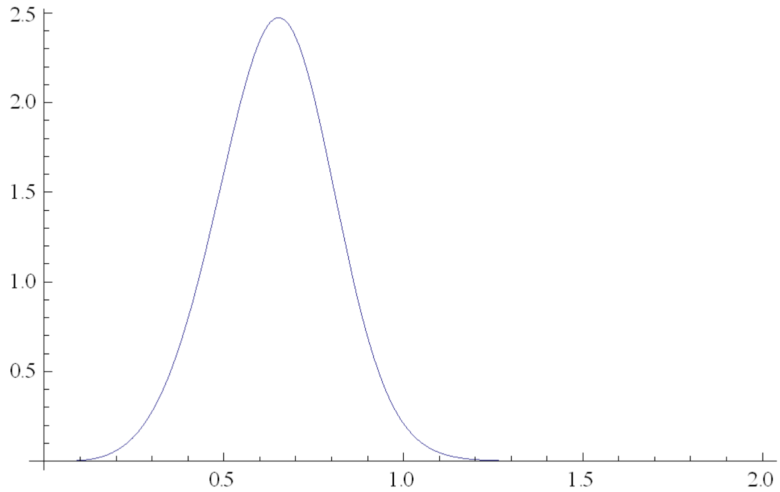

Fig 1. Burr distribution with (c = 4.8773717, τ = 6.157568)

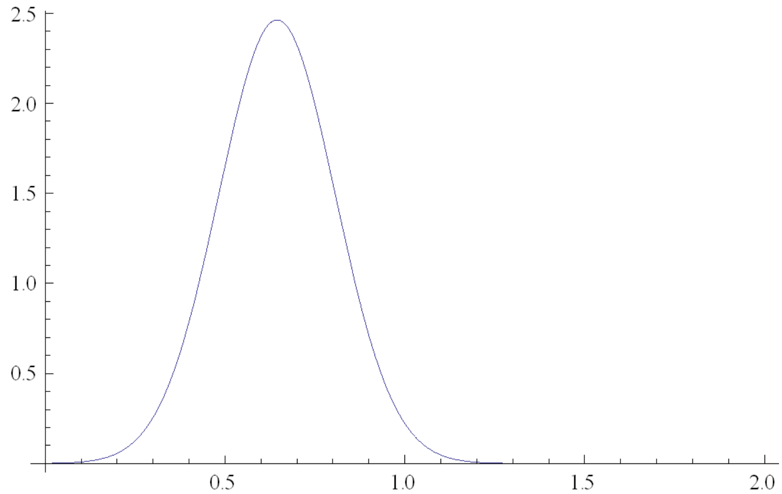

Fig 2. Normal distribution with (μ = 0.644717, σ = 0.16199)

But we know that Tadikamalla (1977) approximate Gamma distribution $\beta = 1$, $\alpha = 2$ with Burr XII with four parameters as $c = 1.39533$, $\rho = 17.97966$, $a = 0.13445$, $b = 15.67968$ **Table 1 and Fig. 1 and 2**. So when X_3~Gamma ($\beta = 1$, $\alpha = 2$) $\Rightarrow X_3$~ Burr IIX with parameter ($c = 1.39533$, $\rho = 17.97966$, $a = 0.13445$, $b = 15.67968$).

Now using Equation 9:

$$\mu_{1:3}^{(k)}[1] = 3\,(15.67968)^k \sum_{s=\max(0,-1)}^{\min(1,0)} C_1 \sum_{l=0}^{s} \binom{s}{l} \sum_{m=0}^{r-s-1} \binom{r-s-1}{m} (-1)^{l+m} \times \sum_{i=0}^{1} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i}$$
$$\beta(\phi - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1) + 17.97966\,(15.67968)^k \sum_{s=\max(0,0)}^{\min(2,0)}$$
$$C_2 \sum_{l=0}^{s} \binom{s}{l} \sum_{m=0}^{r-s-1} \binom{r-s-1}{m} (-1)^{l+m} \times \sum_{i=0}^{1} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i}$$
$$\beta(\phi - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1)$$

$$\Phi = 3(3-1-s+1) + (17.97966)\,(1-1+s+1+m) + 1 = 3(2-s) + (17.97966)(s+m+1) + 1$$

$$\mu_{1:3}^{(k)}[1] = 3\,(15.67968)^k \sum_{s=\max(0,-1)}^{\min(1,0)} C_1 \sum_{l=0}^{s} \binom{s}{l} \sum_{m=0}^{r-s-1} \binom{r-s-1}{m} (-1)^{l+m} \times \sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i}$$
$$\beta(\phi - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1) + 17.97966\,(15.67968)^k \sum_{s=\max(0,0)}^{\min(2,0)}$$
$$C_2 \sum_{l=0}^{s} \binom{s}{l} \sum_{m=0}^{r-s-1} \binom{r-s-1}{m} (-1)^{l+m} \times \sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i}$$
$$\beta(\phi - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1)$$

$$\mu_{1:3}^{(k)}[1] = 3\,(15.67968)^k \sum_{s=0}^{0} C_1 \sum_{l=0}^{s} \binom{s}{l} \sum_{m=0}^{1-s-1} \binom{1-s-1}{m} (-1)^{l+m}$$
$$\times \sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i} \beta(\phi - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1) + 17.97966\,(15.67968)^k$$
$$\sum_{s=0}^{0} C_2 \sum_{l=0}^{s} \binom{s}{l} \sum_{m=0}^{1-s-1} \binom{1-s-1}{m} (-1)^{l+m} \times \sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i}$$
$$\beta(\phi - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1)$$

$$\mu_{1:3}^{(k)}[1] = 3\,(15.67968)^k\, C_1 \sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i}$$
$$\times \beta(6 + (18.97966) - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1) + 17.97966\,(15.67968)^k$$
$$C_2 \sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i} \times \beta(6 + (18.97966) - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1)$$

From (10):

$$C_1 = \frac{2}{1} = 2$$
$$C_2 = \frac{2}{2} = 1$$

$$\therefore\ \mu_{1:3}^{(k)}[1] = 6\,(15.67968)^k \sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i}$$
$$\times \beta(6 + (18.97966) - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1) + 17.97966\,(15.67968)^k$$
$$\sum_{i=0}^{k} \left(\frac{0.13445}{15.67968}\right)^i \binom{k}{i} \times \beta(6 + (18.97966) - \frac{k}{1.39533} + \frac{i}{1.39533} - 1, \frac{k}{1.39533} - \frac{i}{1.39533} + 1)$$

If $k = 1$:

$$\therefore\ \mu_{1:3}[1] = 6\,(15.67968) \sum_{i=0}^{1} \left(\frac{0.13445}{15.67968}\right)^i \binom{1}{i}$$
$$\times \beta(6 + (18.97966) - \frac{1}{1.39533} + \frac{i}{1.39533} - 1, \frac{1}{1.39533} - \frac{i}{1.39533} + 1) + 17.97966\,(15.67968)$$

Table 2. The expected values and the variances in the presence of multiple outlier from Burr XII with four parameters when k = 1, n = 5, c = 1.39533, ρ = 3, a = 0.13445, b = 15.67968, τ = 17.97966

p	r	$\mu_{r:5}[p]$	Variane	p	r	$\mu_{r:5}[p]$	Variane	p	r	$\mu_{r:5}[p]$	Variane
0	1	2.27595	2.68312	1	1	1.41064	0.903247	2	1	1.08231	0.489762
0	2	4.27072	5.49707	1	2	3.08853	3.342250	2	2	2.09302	1.152230
0	3	6.68235	10.71170	1	3	5.37118	8.431080	2	3	3.94371	5.087640
0	4	10.26340	25.48960	1	4	8.86696	22.336100	2	4	7.20476	17.890400
0	5	18.47290	146.95700	1	5	16.83670	133.319000	2	5	14.85890	117.198000

$$\sum_{i=0}^{1}\left(\frac{0.13445}{15.67968}\right)^{i}\binom{1}{i}\times\beta\left(6+(18.97966)-\frac{1}{1.39533}+\frac{i}{1.39533}-1,\frac{1}{1.39533}-\frac{i}{1.39533}+1\right)=1.64008$$

More results can be seen in **Table 2** for n = 5, r = 1,2,…,5 and p = 0,1,2.

Table 1 given below displays the values of the single moments of order statistics in (2.9) when k = 1; n = 5; τ = 17.97966; α = 0.13445; b = 15.67968 and c = 1.39533.

Example (2)

Let X_1, X_2~Barr XII with c = 4.873717, ρ = 5 and X_3~Normal (0.644717, 0.16199). Find $\mu_{1:3}$

Solution

A single outlier in this sample that X_3~Normal (0.644717, 0.16199). But we know that Burr (1942) approximate the normal distribution with Burr XII with two parameters as c = 4.873717 and ρ = 6.157568 from which the more accurate values of μ, σ, $\alpha_3$and α_4 can be obtained as μ = 0.644717, σ = 0.161990, α_3 = 0.00000, α_4 = 0.00000. So when X_3~ Normal (μ = 0.644717, σ = 0.16199) ⇒ X_3~ Burr (c = 4.873717), τ = 6.157568.

Now using Equation 20:

$$\therefore \mu_{1:3}^{(k)}[1]=5\sum_{s=\max(0,-1)}^{\min(1,0)}C_1\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m}\beta\left(\Phi-\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\right)$$
$$+6.157568\sum_{s=\max(0,0)}^{\min(2,0)}C_2\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m}\beta\left(\Phi-\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\right)$$

$$\Phi=5(3-1-s+1)+(6.157568)(1-1+s+1+m)$$
$$+1=5(2-s)+(6.157568)(s+m+1)$$

$$\therefore \mu_{1:3}^{(k)}[1]=5\sum_{s=0}^{0}C_1\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{1-s-1}\binom{1-s-1}{m}(-1)^{l+m}\beta\left(\Phi-\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\right)$$
$$+6.157568\sum_{s=0}^{0}C_2\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{1-s-1}\binom{1-s-1}{m}(-1)^{l+m}\beta\left(\Phi-\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\right)$$

$$\therefore \mu_{1:3}^{(k)}[1]=5\,C_1(1)\,\beta\begin{pmatrix}5(2)+6.157568(1)+1\\ -\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\end{pmatrix}$$
$$+6.157568\;C_2\,(1)\beta\begin{pmatrix}5(2)+6.157568(1)+\\ 1-\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\end{pmatrix}$$

From (10):

$$C_1=\frac{2}{1}=2$$
$$C_2=\frac{2}{2}=1$$

$$\therefore \mu_{1:3}^{(k)}[1]=10\times\beta\begin{pmatrix}5(2)+6.157568(1)+\\ 1-\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\end{pmatrix}$$
$$+6.157568\times\beta\begin{pmatrix}5(2)+6.157568(1)+1\\ -\frac{k}{4.873717}-1,\frac{k}{4.873717}+1\end{pmatrix}$$

If k = 1:

$$\therefore \mu_{1:3}^{(k)}[1]=(10+6.157568)\,\beta\left(17.157568-\frac{1}{4.873717}-1,\frac{1}{4.873717}+1\right)$$
$$=0.522053$$

Table 3. The expected values and the variances in the presence of multiple outlier from Burr XII with two parameters when k = 1, n = 5, c = 4.873717 and ρ = 3, τ = 6.157568

p	r	$\mu_{r:3}[p]$	Variane	p	r	$\mu_{r:3}[p]$	Variane	p	r	$\mu_{r:3}[p]$	Variane
0	1	0.530399	0.0163404	1	1	0.509259	0.0149178	2	1	0.492270	0.01384444
0	2	0.660577	0.0122812	1	2	0.636263	0.0112900	2	2	0.614333	0.01031460
0	3	0.759818	0.0139870	1	3	0.734454	0.0110687	2	3	0.709143	0.01006590
0	4	0.862285	0.0139870	1	4	0.836981	0.0134756	2	4	0.809389	0.01249790
0	5	1.012800	0.0260662	1	5	0.988464	0.0257907	2	5	0.959827	0.02496020

More results can be seen in **Table 3** for n = 5, r = 1,2,…,5 and p = 0,1,2.

1.2. Moments of Order Statistics from Burr XII with Four Parameters Using Theorem of Barakat and Abdelkader (2004)

Let $X_1, X_2, \ldots, X_n$ be independent nonidentically distributed r.v'.s. The k^{th} moment of all order statistics, $\mu_{r:n}^{(k)}$ for $1 \le r \le n$ and k = 1,2,… is given by (Barakat and Abdelkader, 2004) Equation 23:

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} I_j(k) \tag{23}$$

Where Equation 24:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^\infty x^{k-1} \prod_{t=1}^{j} G_{i_t}(x)\,dx, \quad j = 1,2,\cdots,n \tag{24}$$

$G_i(x) = 1 - F_i(x)$, with $(i_1, I_2, \ldots, i_n)$ is a permutation of (1, 2, …, n) for which $i_1 \le i_2 < \ldots < i_n$.

We consider the case when the variable $X_1, X_2, \ldots, X_{n-p}$ are independent observations from Burr XII with four parameters distribution with density Equation 25:

$$f(x) = \frac{\rho c}{b}\left(1+\left(\frac{x-a}{b}\right)^c\right)^{-\rho-1} \left(\frac{x-a}{b}\right)^{c-1}, \quad a \le x \le \infty, \rho > 0, c > 0, b > 0 \tag{25}$$

The corresponding cumulative distribution function F (x) is given as:

$$F(x) = 1-\left(1+\left(\frac{x-a}{b}\right)^c\right)^{-\rho}, \quad x \ge a, \rho > 0, c > 0, b > 0$$

Using (4) in (2):

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_a^\infty x^{k-1}\left(1+\left(\frac{x-a}{b}\right)^c\right)^{-\sum_{t=1}^{j} \rho_{i_t}} dx$$

Substituting $y = \frac{x-a}{b}$:

$$\Rightarrow x = by + a \Rightarrow dx = b\,dy$$

at $x = a \Rightarrow y\ y = 0$, at $x = \infty \Rightarrow y = \infty$

The above Equation reduces to:

$$I_j(k) = k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum b \int_0^\infty (by+a)^{k-1} (1+y^c)^{-\sum_{t=1}^{j} \rho_{i_t}} dy$$

$$= k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum b \int_0^\infty \sum_{m=0}^{k-1} \binom{k-1}{m} a^m (by)^{k-m-1} (1+y^c)^{-\sum_{t=1}^{j} \rho_{i_t}} dy$$

$$k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \sum_{m=0}^{k-1} b^{k-m} \binom{k-1}{m} a^m \int_0^\infty \frac{(y)^{k-m-1}}{(1+y^c)^{-\sum_{t=1}^{j} \rho_{i_t}}} dy$$

Upon using:

$$\int_0^\infty (y)^{k-1} (1+y^c)^{-\alpha}\, dy = \frac{1}{c} \beta\left(\alpha - \frac{k}{c}, \frac{k}{c}\right)$$

where, β (a, b) is the regular beta function:

Table 4. The expected values and the variances in the presence of multiple outlier using Barakat and Abdelkader (2004) and method from Burr XII with four parameters when k=1, n=5, c = 1.39533, ρ = 3, a = 0.13445, b = 15.67968, τ = 17.97966

p	r	$\mu_{r:3}[p]$	Variane	p	r	$\mu_{r:3}[p]$	Variane	p	r	$\mu_{r:3}[p]$	Variane
0	1	2.14150	3.25897	1	1	1.27619	1.24641	2	1	0.947864	0.744643
0	2	4.13627	6.60931	1	2	2.95408	4.13660	2	2	1.958570	1.678890
0	3	6.54790	12.47240	1	3	5.23673	9.83924	2	3	3.809260	6.111950
0	4	10.12900	28.21320	1	4	8.73251	24.68430	2	4	7.070310	19.791600
0	5	18.33840	151.88900	1	5	16.70230	137.81000	2	5	14.724500	121.158000

$$I_j(k) = k \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots$$

$$\sum \sum_{m=0}^{k-1} \frac{b^{k-m}}{c} \binom{k-1}{m} a^m \ \beta\left(\sum_{t=1}^{j} \rho_{i_t} - \frac{k-m}{c}, \frac{k-m}{c}\right)$$

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$

$$\times k \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots$$

$$\sum \sum_{m=0}^{k-1} \frac{b^{k-m}}{c} \binom{k-1}{m} a^m \ \beta\left(\sum_{t=1}^{j} \rho_{i_t} - \frac{k-m}{c}, \frac{k-m}{c}\right)$$

If k = 1:

$$\mu_{r:n} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$

$$\times \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots \sum \frac{b}{c} \beta\left(\sum_{t=1}^{j} \rho_{i_t} - \frac{1}{c}, \frac{1}{c}\right)$$

Example

Let X_1, X_2~Barr XII with c = 1.39533, ρ = 3, a = 0.13445, b = 15.67968 and X_3~ Gamma (β = 1, α = 2). Find $\mu_{1:3}$.

Solution

X_3~ Gamma (β = 1, α = 2) ⇒ X_3~Burr IIX with parameters (c = 1.39533, ρ = 17.97966, a = 0.13445, b = 15.67968):

$$\mu_{1:3} = \sum_{j=3}^{3} (-1)^{j-3} \binom{j-1}{2}$$

$$\sum_{1\le i_1<i_2<\cdots<i_j\le 3} \cdots$$

$$\sum \frac{b}{c} \beta\left(\sum_{t=1}^{j} \rho_{i_t} - \frac{1}{c}, \frac{1}{c}\right)$$

$$= \sum_{1\le i_1<i_2<\cdots<i_3\le 3} \cdots$$

$$\sum \frac{b}{c} \beta\left(\sum_{t=1}^{3} \rho_{i_t} - \frac{1}{c}, \frac{1}{c}\right)$$

$$= \frac{b}{c} \beta\left(\rho_1 + \rho_2 + \rho_3 - \frac{1}{c}, \frac{1}{c}\right)$$

$$= \frac{15.67968}{1.39533} \beta\left(3 + 3 + 17.97966 - \frac{1}{1.39533}, \frac{1}{1.39533}\right)$$

$$= 1.50563$$

More results can be seen in **Table 4** for r = 1,2,…,5 and p = 0,1,2.

2. CONCLUSION

We can find moments of order statistics from independent and nonidentically distributed random variables for any distribution with no simple closed from using approximating idea with Burr XII or any other distribution.

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