Original Research Paper

Convergence of Renormalization Group Transformations of Gibbs Random Field

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Abstract: Statistical mechanics describes interaction between particles of a physical system. Particle properties of the system can be modelled with a random field on a lattice and studied at different distance scales using renormalization group transformation. Here we consider a thermodynamic limit of a lattice model with weak interaction and we use semi-invariants to prove that random fields transformed by renormalization group converge in distribution to an independent field with Gaussian distribution as the distance scale infinitely increases; it is a generalization of the central limit theorem to weakly dependent fields on a lattice.

Keywords: Gibbs Measure, Renormalization Group, Semi-Invariant, Thermodymanic Limit, Weak Dependence

Introduction

The classical Central Limit Theorem (CLT) considers a sequence of independent random variables and their normalized sums. Here we consider a sequence of weakly dependent random fields on a multi-dimensional integer lattice. We are interested in the limiting distribution of normalized sums of these variables, similar to the sums in the classical CLT. Such problems arise in the research of Renormalization Group (RG) in statistical mechanics.

The concept of RG as a scale transformation was introduced and studied in works of Kadanoff (Kadanoff, 1966; Kadanoff, 2013), Wilson and Kogut (1974). Originally RG was defined in terms of Hamiltonian (interpreted as the interaction potential). A rigorous formula of the renormalized Hamiltonian was derived by Kashapov (1980). Bertini *et al.* (1999) and Lorinczi *et al.* (1998) studied Gibbs property of the renormalized Hamiltonian.

Other research on RG are based on limit theorems of probability theory. Sinai (1976) studied distributions invariant under the RG transformation and showed that Gaussian distribution is one of them. Newman (1980) proved the CLT on an integer lattice under Fortuin-Kasteleyn-Ginibre (FKG) conditions. Bolthausen (1982) proved the CLT on an integer lattice under some strong conditions.

In this paper we study the limiting distribution of Gibbs random field under the RG transformations and we improve our results from (Kachapova and Kachapov, 2015). We show that under the condition $|\lambda| < C$ the limiting distribution in a high-temperature region is an independent Gaussian distribution. The novelty of our result is in finding a broad condition for the interaction parameter λ, for which the CLT on a lattice holds; this condition is $|\lambda| < C$ for a constant C depending only on the lattice dimension. This is a simple condition and is easy to check and it is stated in a form preferable for physicists, without tedious technical details.

The FKG conditions in the Newman's version of CLT (Newman, 1980) do not always hold; for example, they hold for the ferromagnetic Ising model but not for the anti-ferromagnetic one; while our theorem covers both models and more.

The conditions in the Bolthausen's theorem (1982) involve supremum of probabilities and supremum of covariances, which are difficult to estimate. Also Bolthausen proved his theorem under the assumption of absolute convergence of three series and positivity of a fourth series, while in our paper we prove convergence of all necessary series. In his theorem Bolthausen did not consider RG transformations but only normalized sums of a random variable on finite sets and he proved convergence in distribution of these sums to a single random variable. In our paper we prove convergence in distribution of RG transformations of a random field to another random field.

Mathematicians doing research in statistical mechanics try to create the mathematical structures that make foundation of physical theories. They appreciate rigorous, non-contradictory and transparent theories.

They also make effort to obtain simplest possible proofs for existing theorems. In our paper we use a new approach in proving the CLT for weakly dependent random fields; this approach is based on estimation of semi-invariants.

We apply the techniques of Malyshev and Minlos (1991; Malyshev, 1980) to estimate semi-invariants of a random field and we use these estimations to prove a generalization of the CLT to weakly dependent random fields on a lattice.

Semi-invariants are synonyms for cumulants and Ursell functions. We give the definition of semi-invariants and briefly describe their properties in section 2. In section 2 we also introduce other necessary concepts from probability theory and statistical mechanics and briefly prove some relevant lemmas.

In section 3 we state the main result of this paper: the central limit theorem for Gibbs random field transformed by RG, with a brief discussion of its meaning.

The rest of the paper develops techniques for proving the main theorem. In particular, in section 4 we prove an inequality about the number of links in a set with a symmetric binary relation and apply it to estimate semi-invariants of a random field (Estimation Theorem).

In section 5 we prove the main theorem. In subsections 5.1 and 5.2 we prove a series of lemmas, which lead to the direct proof of the main theorem in subsections 5.3 and 5.4. In particular, we find an expression for the limiting variance and show the equality to 0 of all other limiting semi-invariants of the random field transformed by RG. We complete the proof of the main theorem by applying Carleman theorem to the limiting distribution.

2. Main Concepts

2.1. Semi-Invariants

Denote $E(X)$ the expectation of a random variable X. Semi-invariant is a generalization of the concepts of expectation and covariance. The following is a slight modification of the definition in (Malyshev and Minlos, 1991), pg. 27-33.

Definition 2.1

Suppose $X_1,..., X_m$ are random variables on the same probability space and $M = \{1, 2,..., m\}$ is the set of their indices. For any $S \subseteq M$, we denote $X_S = \prod_{i \in S} X_i$. We assume that the expectation of every such product is finite.

A **semi-invariant** of random variables $X_1,..., X_m$ is:

$$\langle X_1,..., X_m \rangle = \sum_{\alpha} (-1)^{k-1} (k-1)! \, E(X_1)...E(X_m),$$

where the sum is taken over all partitions $\alpha = \{S_1,..., S_k\}$ of the set M. By a partition we mean a set of

disjoint, non-empty subsets of M such that their union equals M.

Notation

If $I = (i_1,..., i_m)$ is a sequence or a set of indices, we denote $\left\langle X_I^{\backslash} \right\rangle = \left\langle X_{i_1},..., X_{i_m} \right\rangle$.

Semi-invariants characterize the distribution and dependence of random variables. Other terms for a semi-invariant are *cumulant* and *Ursell function*.

Example 2.1

Suppose X, X_1, X_2 and X_3 are random variables. Denote μ the expectation of X and σ the standard deviation of X. Then the following hold:

1) $\langle X \rangle = \mu$.
2) $\langle X_1, X_2 \rangle = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle = cov\,(X_1, X_2)$, the covariance of X_1 and X_2.
3) $\langle X_1, X_2, X_3 \rangle = \langle X_1 X_2 X_3 \rangle - \langle X_1 \rangle \langle X_2 X_3 \rangle - \langle X_2 \rangle \langle X_1 X_3 \rangle - \langle X_3 \rangle \langle X_1 X_2 \rangle + 2\langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle$.
4) $\langle X, X \rangle = \sigma^2$, the variance of X.
5) $\langle X, X, X \rangle / \sigma^3$ equals the skewness of X.
6) $\langle X, X, X, X \rangle / \sigma^4$ equals the kurtosis of X.

Lemma 2.1

1. A semi-invariant is a symmetrical and multi-linear functional on random variables
2. If $0 < n < m$ and two random vectors $(X_1,..., X_n)$ and $(X_{n+1},..., X_m)$ are independent of each other, then $\langle X_1,..., X_n, X_{n+1},..., X_m \rangle = 0$.

3. For set $M = \{1, 2,..., m\}$:
$$E\left(X_M\right) = \left\langle X_M \right\rangle = \sum_{\alpha} \left\langle X_{S_1}^{\backslash} \right\rangle ... \left\langle X_{S_k}^{\backslash} \right\rangle ,$$

where the sum is taken over all partitions $\alpha = \langle S_1,..., S_k \rangle$ of the set M.
4. If $m > 1$ and at least one of random varables $X_1,..., X_m$ is constant, then $\langle X_1,..., X_m \rangle = 0$.

Proof can be found in (Malyshev and Minlos, 1991).
The following is a well-known lemma about semi-invariants of normal distribution.

Lemma 2.2

Suppose random variables Y_1, $Y_2,..., Y_m$ have an independent multivariate normal distribution and $M = \{1, 2,..., m\}$ is the set of their indices.

1. If $k \geq 3$ and $i_1,..., i_k \in M$, then $\left\langle Y_{i_1},..., Y_{i_k} \right\rangle = 0$.

2. If $i, j \in M$ and $i \neq j$, then $\langle Y_i, Y_j \rangle = 0$.

Definition 2.2

We say that random variables $Y_1,..., Y_m$ satisfy the **m-variate Carleman** condition if:

$$\sum_{n=1}^{\infty} (M_{2n})^{-\frac{1}{2n}} = \infty, \text{ where } M_k = \sum_{i=1}^{m} \langle Y_i^k \rangle. \qquad (1)$$

For example, if X is bounded, then it satisfies Carleman condition. The logarithmic normal distribution does not satisfy Carleman condition and is not defined uniquely by its moments. In order to prove that random variables $X_1, X_2,..., X_m$ with identical distribution satisfy the m-variate Carleman condition, it is sufficient to check the 1-variate Carleman condition for only one of the random variables.

We will use the following version of Carleman theorem.

Theorem 2.1. (Carleman theorem)

Suppose random vectors $(X_1,..., X_m)$ and $(Y_1,..., Y_m)$ have equal corresponding moments and the variables $Y_1,..., Y_m$ satisfy the m-variate Carleman condition. Then the random vectors $(X_1,..., X_m)$ and $(Y_1,..., Y_m)$ have the same probability distribution.

Lemma 2.3

If a random variable Z has the standard normal distribution, then it satisfies the 1-variate Carleman condition.

Proof

Clearly, $\langle Z^{2k} \rangle = (2k-1)!!$ If integer $k \geq 1$, then $(2k-1)!! \leq k^k$. Therefore

$$\sum_{k=1}^{\infty} \langle Z^{2k} \rangle^{-\frac{1}{2k}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty.$$

Similarly it can be proven that a random variable with exponential distribution satisfies the 1-variate Carleman condition.

Lemma 2.4

Suppose random variables $X_1, X_2, ..., X_m$ are uncorrelated, identically distributed and satisfy the following condition:

$$\langle X_{i_1},..., X_{i_m} \rangle = 0 \text{ for } k \geq 3; \ 1 \leq i_1 ,..., i_m \leq m.$$

Then $X_1, X_2, ..., X_m$ are independent and have a multivariate normal distribution.

Proof

Denote $\mu = \langle X_i \rangle$, $\sigma^2 = \langle X_i, X_i \rangle$. Consider independent random variables $Z_1, Z_2, ..., Z_m$, where each Z_i has the standard normal distribution and denote $Y_i = \sigma Z_i$. Then $Y_1,..., Y_m$ are independent and have a multivariate normal distribution.

By Lemma 2.3 each Z_i satisfies the 1-variate Carleman condition and so does each Y_i. Since $Y_1,..., Y_m$ are identically distributed, they satisfy the m-variate Carleman condition.

By Lemma 2.2, the random vectors $(Y_1, ..., Y_m)$ and $(X_1-\mu ,..., X_m-\mu)$ have the same corresponding semi-invariants and the same corresponding moments, since semi-invariants uniquely determine moments. So by Carleman theorem, the random vectors $(Y_1, ..., Y_m)$ and $(X_1-\mu ,..., X_m-\mu)$ have the same distribution, which is an independent multivariate normal distribution. Hence the random vector $(X_1, ..., X_m)$ has an independent multivariate normal distribution.

2.2. Interaction Model in Statistical Mechanics

For the rest of the paper we fix a natural number $v \geq 1$ and consider a v-dimensional integer lattice:

$$\mathbb{Z}^v = \{(t_1,..., t_v) \mid t_i \in \mathbb{Z}, i = 1, 2,..., v\}$$

with the distance between any two points s and t defined by:

$$\|s - t\| = \sum_{i=1}^{v} |s_i - t_i|.$$

Denote $\overline{0} = (0,...,0)$, the origin. Fix a set $D \subseteq \mathbb{R}$ with at least 2 elements and denote $\Omega = \{\omega \mid \omega: \mathbb{Z}^v \to D\}$. An element ω of Ω is called a *configuration* and is interpreted as a state of a physical system in statistical mechanics.

For each $t \in \mathbb{Z}^v$ a function $X_t : \Omega \to D$ is defined by the following:

$$X_t(\omega) = \omega(t).$$

We define Σ as the σ-algebra generated by sets of the form $\{\omega \in \Omega \mid \omega(t) \leq a\}$ for all $t \in \mathbb{Z}^v$ and $a \in D$. We fix a probability measure P_0 on (Ω, Σ) such that:

for any $a \in D$, $P_0(\omega(t) < a)$ does not depend on t $\qquad$ (2)
and
for any $a_1,..., a_n \in D$ and distinct $t_1,..., t_n \in \mathbb{Z}^v$:
$$P_0(\omega(t_1) < a_1,..., \omega(t_n) < a_n)$$
$$= P_0(\omega(t_1) < a_1) \cdot ... \cdot P_0(\omega(t_n) < a_n). \qquad (3)$$

Then $\{X_t \mid t \in \mathbb{Z}^v\}$ is an independent random field on the probability space (Ω, Σ, P_0) and this field is translation invariant. Clearly, the random variables X_t, $t \in \mathbb{Z}^v$, are identically distributed with respect to the measure P_0.

We also assume that the following conditions are satisfied:

$$each\ X_t\ has\ a\ finite\ moment\ of\ m^{th}\ order, m = 1, 2, 3,...; \qquad (4)$$

each X_t satisfies the 1-variate Carleman condition:

$$\sum_{n=1}^{\infty}\left(\left\langle X_t^{2k}\right\rangle_0\right)^{-\frac{1}{2k}} = \infty. \qquad (5)$$

We denote $\langle\cdot\rangle_0$ the expectation with respect to the measure P_0.

Note 1

There always exists a probability measure P_0 satisfying (2) - (3) and for which $\{X_t \mid t \in \mathbb{Z}^v\}$ satisfies the conditions (4) - (5). Here is an example. Let $F(x)$ be a probability distribution function satisfying Carleman condition, that is:

$$\sum_{n=1}^{\infty}\left(M_{2n}\right)^{-\frac{1}{2n}} = \infty, where\ M_k = \int_{-\infty}^{+\infty} x^k dF(x).$$

As mentioned before, the normal and exponential distributions are some of the distributions satisfying Carleman condition.

Probability measure P_0 is defined by:

$$P_0(\omega(t) < a) = F(a)\ and\ equality\ (3).$$

Then the conditions (2) - (5) are satisfied.

We fix an increasing sequence Λ_N of finite subsets of $\mathbb{Z}^v$ such that $\Lambda_N \subset \Lambda_{N+1}$ for any $N \in \mathbb{N}$ and

$$\bigcup_{N=1}^{\infty}\Lambda_N = \mathbb{Z}^v.$$

Denote $R = \{\{s, t\}\mid s, t \in \mathbb{Z}^v\ and\ \|s-t\| = 1\}$. R is the set of all pairs of neighbouring nodes in the lattice $\mathbb{Z}^v$. Denote $RB = \{u \in \mathbb{Z}^v \mid$ one coordinate of u is 1 and the others are 0$\}$. RB is the standard basis in $\mathbb{R}^v$.

Definition 2.3

Interaction model is defined by a triple of objects (N, λ, φ), where
(i) $N \in \mathbb{N}, N \geq 1$;
(ii) $\lambda \in \mathbb{R}$;
(iii) for any $u \in RB$, $\varphi_u: \mathbb{R}^2 \to \mathbb{R}$ is a Borel function such that $|\varphi_u| \leq 1$.

The interaction model includes a set Λ_N (as defined before), **potential** Φ and **interaction energy** U_N defined as follows.

1) For any $B \in R$ we define a random variable Φ_B on the probability space (Ω, Σ, P_0). Any $B \in R$ has the form $B = \{r, r + u\}$, where $u \in RB$, so we define:

$$\Phi_B(\omega) = \varphi_u\left(X_r(\omega), X_{r+u}(\omega)\right).$$

Such Φ_B represents interaction between neighbours r and $r + u$.

2) Function $U_N: \Omega \to R$ is defined by the following:

$$U_N(\omega) = -\lambda \sum_{B \in R, B \subset \Lambda_N} \Phi_B(\omega). \qquad (6)$$

$U_N(\omega)$ characterizes the energy of configuration ω in Λ_N.

This completes the definition of the interaction model.

Note 2

In the interaction model a union of the random fields $\{X_t \mid t \in \mathbb{Z}^v\}$ and $\{\Phi_B \mid B \in R\}$ is translation invariant. This means: for any $t_1,..., t_m, r \in \mathbb{Z}^v$ and any $B_1,..., B_n \in R$ the random vectors $\left(X_{t_1},...,X_{t_m},\Phi_{B_1},...,\Phi_{B_n}\right)$ and $\left(X_{t_1+r},...,X_{t_m+r},\Phi_{B_1+\bar{r}},...,\Phi_{B_n+\bar{r}}\right)$ have the same distribution (here for $B_i = \{s_i, t_i\}$, $B_i + \bar{r}$ denotes $\{s_i + r, t_i + r\}$).

The interaction model describes a physical system with many particles represented by points of the set Λ_N in the integer lattice. The random field X_t describes some property of the physical system. The function U_N characterizes the interaction energy of the system and $|\lambda|$ is proportional to the inverse temperature of the system. The parameter λ also characterizes the strength of interaction between particles and we assume that only neighbouring particles interact.

Example 2.2

The statistical model with $\lambda = 0$ describes a physical system with no interaction between its elements, e.g., ideal gas.

Example 2.3

Potts model with parameter q ($q \in \mathbb{N}, q \geq 1$).
It is a particular case of interaction model, where $D = \{1, 2,..., q\}$ and the probability measure P_0 is defined by:

$$P_0(\omega(t) = i) = \frac{1}{q},\ i = 1, 2,..., q,\ and\ equality\ (3).$$

Since D is finite, each X_t satisfies the Carleman condition. The rest of the model is defined by:

$$\varphi_u(x,y) = \delta(x,y), where\ \delta(x,y) = \begin{cases} 1 & if\ x = y, \\ 0 & if\ x \neq y. \end{cases}$$

$$So\ \Phi_{\{s,t\}}(\omega) = \delta(X_s(\omega), X_t(\omega))\ and$$

$$U_N(\omega) = -\lambda \sum_{\{s,t\} \in R; s,t \in \Lambda_N} \delta(X_s(\omega), X_t(\omega))$$

2.3. Gibbs Modification

Gibbs modification was introduced in (Malyshev and Minlos, 1991). Since it is important for our paper, we also provide the definition.

Definition 2.4

For the interaction model given by (N, λ, φ) we define the **associated probability space** $(\Omega, \Sigma_N, P_{\lambda,N})$ as follows:

1. The sample space is the set Ω as defined before.
2. Σ_N is the sigma-algebra generated by X_t, $t \in \Lambda_N$.
3. For any event $A \in \Sigma_N$ the probability is defined by:

$$P_{\lambda,N}(A) = \frac{\left\langle I_A e^{-U_N} \right\rangle_0}{\left\langle e^{-U_N} \right\rangle_0},$$

where I_A denotes the indicator of event A.

The probability measure $P_{\lambda,N}$ is called **(finite) Gibbs modification** on Λ_N. This completes the definition.

Note: Clearly, $P_{\lambda,N}(\Omega) = \frac{\left\langle 1 \cdot e^{-U_N} \right\rangle_0}{\left\langle e^{-U_N} \right\rangle_0} = 1$.

Example 2.4

Ising model with parameters N, λ and h is usually defined with probabilities:

$$P(\omega(t) = 1) = P(\omega(t) = -1) = \frac{1}{2}$$

and potential $W_N = U_N + V_N$, where

$$U_N = -\lambda \sum_{\{s,t\} \in R, s,t \in \Lambda_N} X_s(\omega) X_t(\omega)\ \ and\ \ V_N = -\lambda \sum_{s \in \Lambda_N} X_s(\omega).$$

Σ_N is defined in the same way as in the interaction model and probability measure P_G is defined by:

$$P_G(A) = \frac{\left\langle I_A e^{-W_N} \right\rangle_0}{\left\langle e^{-W_N} \right\rangle_0}\ \ for\ any\ A \in \Sigma_N,$$

where the expectations are with respect to the probability measure P.

We can define the Ising model as a particular case of interaction model. We take $D = \{-1, 1\}$; P_0 is defined by:

$$P_0(\omega(t) = a) = \frac{e^{-ha}}{e^{-h} + e^{h}}\ \ (a = \pm 1)\ and\ equality\ (3);$$

$$\varphi_u(x,y) = xy.$$

So $\Phi_{\{s,t\}}(\omega) = X_s(\omega) X_t(\omega)$ and

$$U_N = -\lambda \sum_{\{s,t\} \in R, s,t \in \Lambda_N} X_s(\omega) X_t(\omega).$$

Then we show that the probability space $(\Omega, \Sigma_N, P_{\lambda,N})$ is the same as (Ω, Σ_N, P_G) in the usual definition of the Ising model.

Proof that $P_G = P_{\lambda,N}$.
Fix $A \in \Sigma_N$.

$$\left\langle I_A e^{-W_N} \right\rangle = \sum_{\omega \in \Omega} I_A(\omega) e^{-W_N(\omega)} P(\omega)$$

$$= \sum_{\omega \in A} e^{-U_N(\omega) - V_N(\omega)} \frac{1}{2^{|\Lambda_N|}}$$

$$= \frac{1}{2^{|\Lambda_N|}} \sum_{\omega \in A} e^{-U_N(\omega)} e^{-h \sum_{s \in \Lambda_N} X_s(\omega)}$$

$$= \frac{1}{2^{|\Lambda_N|}} \sum_{\omega \in A} \left(e^{-U_N} \prod_{s \in \Lambda_N} e^{-hX_s(\omega)} \right)$$

$$= \left(\frac{1}{2} \right)^{|\Lambda_N|} \left(\frac{e^{-h} + e^{h}}{e^{-h} + e^{h}} \right)^{|\Lambda_N|} \sum_{\omega \in A} \left(e^{-U_N(\omega)} \prod_{s \in \Lambda_N} e^{-h\omega(s)} \right)$$

$$= \left(\frac{e^{-h} + e^{h}}{2} \right)^{|\Lambda_N|} \sum_{\omega \in A} \left(e^{-U_N(\omega)} \prod_{s \in \Lambda_N} \frac{e^{-h\omega(s)}}{e^{-h} + e^{h}} \right)$$

$$= \left(\frac{e^{-h} + e^{h}}{2} \right)^{|\Lambda_N|} \sum_{\omega \in A} e^{-U_N(\omega)} P_0(\omega)$$

$$= \left(\frac{e^{-h} + e^{h}}{2} \right)^{|\Lambda_N|} \left\langle I_A e^{-U_N} \right\rangle_0$$

Similarly, $\left\langle e^{-W_N} \right\rangle = \left(\frac{e^{-h} + e^{h}}{2} \right)^{|\Lambda_N|} \left\langle e^{-U_N} \right\rangle_0$.

Therefore

$$P_G(A) = \frac{\left\langle I_A e^{-W_N} \right\rangle}{\left\langle e^{-W_N} \right\rangle} = \frac{\left\langle I_A e^{-U_N} \right\rangle_0}{\left\langle e^{-U_N} \right\rangle_0} = P_{\lambda,N}(A).$$

Lemma 2.5

Suppose $\lambda = 0$. Then the following hold.
1. $P_{0,N} = P_0$ on Σ_N.

2. $\{X_t \mid t \in \Lambda_N\}$ is an independent random field with respect to $P_{0,N}$.

3. The distribution of the random field $\{X_t \mid t \in \Lambda_N\}$ with respect to Gibbs modification $P_{0,N}$ does not depend on N.

4. Suppose $T = \{t_1,\dots, t_n\} \subset \Lambda_N$, $S = \{s_1,\dots, s_m\} \subset \Lambda_N$ and $T \cap S = \varnothing$. Suppose F_i ($i = 1,\dots, k$) are random variables dependent on $X_{t_1},\dots,X_{t_n}$ and G_j ($j = 1,\dots, p$) are random variables dependent on $X_{s_1},\dots,X_{s_m}$. Then the random vectors $(F_1,\dots, F_k)$ and $(G_1,\dots, G_p)$ are independent of each other with respect to P_0.

Proof

Let $\lambda = 0$. Then $U_N = 0$.

1. For any $A \in \Sigma_N$, $P_{0,N}(A) = \dfrac{\left\langle I_A e^0 \right\rangle_0}{\left\langle e^0 \right\rangle_0} = \left\langle I_A \right\rangle_0 = P_0(A)$.

2. This part holds because the random variables X_t, $t \in \Lambda_N$, are independent with respect to the measure P_0.

3. Follows from part 1.

4. Follows from part 2.

2.4. Gibbs Measure and Thermodynamic Limit

The definition of Gibbs measure is given in (Dobrushin, 1968); it is a probability measure on (Ω, Σ). For our results it is sufficient to consider Gibbs measure P_λ as the limit of Gibbs modifications $P_{\lambda,N}$ as $N \to \infty$:

$$P_\lambda(A) = \lim_{N \to \infty} P_{\lambda,N}(A) \quad \text{for any } A \in \Sigma. \tag{7}$$

Malyshev and Minlos (1991) established necessary and sufficient conditions when the equality (7) holds; the results in (Kashapov, 1980) imply that the equality (7) holds for all λ with $|\lambda| < C$, where C is the constant from our main theorem (Theorem 3.1).

Let us see what happens to the interaction model and associated probability space when $N \to \infty$. Clearly, the finite set Λ_N transforms into the lattice $\mathbb{Z}^v$, Σ_N transforms into Σ and the Gibbs modification $P_{\lambda,N}$ transforms into the Gibbs measure P_λ.

Definition 2.5

The **thermodynamic** or **macroscopic limit** of interaction model is the lattice $\mathbb{Z}^v$ together with the limiting probability space $(\Omega, \Sigma, P_\lambda)$.

Clearly, $\{X_t \mid t \in \mathbb{Z}^v\}$ is a random field on the limiting probability space. We denote $\langle \cdot,\dots,\cdot \rangle_\lambda$ semi-invariants with respect to the Gibbs measure P_λ.

2.5. Renormalization Group

The following concept was introduced by Kadanoff (1966).

Definition 2.6

Fix a natural number $k > 1$ and a real number $\alpha \geq v$.

For each $r = (r_1, r_2,\dots, r_v) \in \mathbb{Z}^v$ consider a cube C_r^k of side length k with vertex kr:

$$C_r^k = \{ t \in \mathbb{Z}^v \mid kr_i \leq t_i < k(r_i + 1), i = 1, 2,\dots, v\}.$$

A **renormalization group** (RG) with parameters k and α is a transformation that assigns to each random field $\{Z_t \mid t \in \mathbb{Z}^v\}$ a new random field $\{ Y_r^{(k)} \mid r \in \mathbb{Z}^v \}$ given by:

$$Y_r^{(k)} = k^{-\frac{\alpha}{2}} \sum_{t \in C_r^k} \left(Z_t - \left\langle Z_t \right\rangle_\lambda \right).$$

RG is a scaling transformation. It allows to study a physical system at different distance scales, such as atomic and molecular levels. Details of its physical interpretation can be found in (Kadanoff, 2013).

We are interested in the distribution of the result $Y_r^{(k)}$ of the RG transformation of the field $\{X_t \mid t \in \mathbb{Z}^v\}$.

3. The Central Limit Theorem for the Field X_t Transformed by RG

Theorem 3.1. (Main Theorem)

Consider the thermodynamic limit of interaction model with parameter λ. Suppose a renormalization group with parameters k and α transforms the random field $\{X_t \mid t \in \mathbb{Z}^v\}$ into a random field $\{ Y_i^{(k)} \mid r \in \mathbb{Z}^v \}$.

There exists a positive constant C such that for any $|\lambda| < C$ the following hold.

1. Suppose $\alpha > v$. Then the field $Y_r^{(k)} \to 0$ in mean square as $k \to \infty$.

2. Suppose $\alpha = v$. Then as $k \to \infty$, the random field $\{ Y_i^{(k)} \mid r \in \mathbb{Z}^v \}$ converges in distribution to an independent random field with Gaussian distribution (i.e., any finite subset of the field has a multivariate normal distribution). Each of the variables of the limiting field has 0 expectation and the positive variance given by:

$$V = \sum_{n=0}^{\infty} \lambda^n V_n ,$$

where each coefficient V_n is a finite sum of semi-invariants of X_t and Φ_B with respect to P_0 ($t \in \mathbb{Z}^v$, $B \in R$). Exact formula for coefficients V_n is formula (27) in subsection 5.4.

Proof is given in section 5.

This theorem can be considered as a generalization of the classical Central Limit Theorem (CLT). Instead of a sequence of independent random variables we have a weakly dependent random field $\{X_t \mid t \in \mathbb{Z}^v\}$. It is weakly dependent because $|\lambda|$ is small and λ characterizes the strength of interaction.

The classical CLT considers a sequence of independent identically distributed random variables with finite variances and states that their normalized sums converge in distribution to a normal random variable. Theorem 3.1.2) also states convergence in distribution and that the limiting distribution is normal but in this case it is the distribution of an independent normal field.

In other words, Theorem 3.1 states: in systems with weak interaction the distribution of the normalized sums over big regions is approximately independent and normal.

4. Estimation of Dependencies

4.1. Estimation Theorem

The proof of the main theorem in Section 5 is based on estimations of semi-invariants. In this section we prove an inequality (Theorem 4.1), which will be applied to estimating semi-invariants. This was inspired by Estimates of Intersection Number in (Malyshev and Minlos, 1991). Here we have improved our estimate from (Kachapova and Kachapov, 2015) and simplified the proof.

In this section we consider a countable set $\mathfrak{A}$ with a reflexive, symmetric binary relation. If elements a, b of $\mathfrak{A}$ are in this relation, we say that a and b are linked. Thus, any element of $\mathfrak{A}$ is always linked to itself (reflexivity). If a is linked to b, then b is linked to a (symmetry).

Denote $l(a)$ the number of elements in $\mathfrak{A}$ that are linked to a. In this section we assume that there is a constant L such that $l(a) \leq L$ for all $a \in \mathfrak{A}$.

Definition 4.1

For any sequence $\alpha = (a_1,..., a_m)$ of elements of $\mathfrak{A}$ we use the following notations:

1. $J(\alpha) = \{i_1,..., i_k\}$ is the set of indexes of elements of α when the elements are written without repetition.
2. $n_i(\alpha)$ is the number of elements of α that equal a_i ($i = 1,..., m$). This number is called the **multiplicity** of a_i.
3. $\upsilon_i(\alpha)$ is the number of elements of α that are linked to a_i ($i = 1,..., m$).

Theorem 4.1. (Estimation Theorem)

For any sequence α of elements of $\mathfrak{A}$:

$$\frac{1}{|\alpha|} \sum_{j \in J(\alpha)} n_j \, ln\left(\frac{\upsilon_j}{n_j}\right) \leq ln \, L \, , \qquad (8)$$

where $|\alpha| = m$ is the length of the sequence α, $n_j = n_j(\alpha)$ and $\upsilon_j = \upsilon_j(\alpha)$.

This estimate cannot be improved.

Proof

Fix a sequence $\alpha = (a_1,..., a_m)$ of elements of $\mathfrak{A}$. Denote $J = J(\alpha)$ for brevity. We define a link matrix d_{ij} as follows:

$$d_{ij} = \begin{cases} 1 & \text{if } a_i \text{ is linked to } a_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $j = 1,..., m$:

$$\sum_{i \in J} d_{ij} \leq l\left(a_j\right) \leq L \text{ and}$$

$$\sum_{i \in J} \upsilon_i = \sum_{i \in J}\sum_{j \in J} d_{ij} n_j = \sum_{j \in J} n_j \sum_{i \in J} d_{ij} \leq L \sum_{j \in J} n_j \, .$$

Thus,

$$\sum_{i \in J} \upsilon_i \leq mL \, . \qquad (9)$$

Next we use Jensen inequality, which states for a concave function f and numbers $x_1,..., x_n$ in its domain:

$$\frac{1}{n}\sum_{i=1}^{n} f\left(x_i\right) \leq f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

We apply the Jensen inequality to the concave function logarithm and $x_i = \dfrac{\upsilon_i}{n_i}$ ($i = 1, ..., m$):

$$\frac{1}{m}\sum_{j \in J} n_j \, ln\left(\frac{\upsilon_j}{n_j}\right) = \frac{1}{m}\sum_{i=1}^{m} ln\left(\frac{\upsilon_i}{n_i}\right)$$

$$\leq ln\left(\frac{1}{m}\sum_{i=1}^{m}\frac{\upsilon_i}{n_i}\right) \leq ln\left(\frac{1}{m}\sum_{j \in J} n_j \frac{\upsilon_j}{n_j}\right)$$

$$= ln\left(\frac{1}{m}\sum_{j \in J}\upsilon_j\right) \leq ln\left(\frac{1}{m} mL\right) = ln \, L \text{ by (9).}$$

The following example shows that the estimate cannot be improved. $\mathfrak{A}$ is an arbitrary countable set and m is any positive integer. We take $\alpha = (a_1,..., a_m)$, where all a_i are distinct. We assume that any a_i and a_j are linked ($i, j = 1,..., m$) and there are no other links in $\mathfrak{A}$. Then for each $i = 1,..., m$, $n_i(\alpha) = 1$, $\upsilon_i(\alpha) = l(a_i) = m$ and $L = m$. The left-hand side of (8) is:

$$\frac{1}{m}\sum_{j \in J} 1 \cdot \ln m = \frac{1}{m} \cdot m \cdot \ln m = \ln m = \ln L \, .$$

4.2. Application of the Estimation Theorem to Semi-Invariants

First we introduce some notations. We take $\mathfrak{A} = R \cup \{\{t\} \mid t \in \mathbb{Z}^v\}$. Two subsets T and S of $\mathbb{Z}^v$ are said to be linked if $T \cap S \neq \varnothing$. By Lemma 2.5.4), if sets T and S are not linked, then they correspond to independent random vectors. So links correspond to possible dependencies of random vectors.

Any element of the form $\{t\}$ is linked to itself and to $2v$ elements of the form $\{t, r\}$, so $l(\{t\}) = 2v + 1$. Any element of the form $\{r, s\}$ is linked to elements $\{r\}$, $\{s\}$, $2v$ elements of the form $\{r, t\}$ and $2v$ elements of the form $\{s, t\}$, so the total is $4v + 1$ (because the element $\{r, s\}$ is counted twice). Then $L = 4v + 1$ (each element of $\mathfrak{A}$ is linked to at most $4v + 1$ elements).

Definition 4.2

1. A **family** (of elements of the set $\mathfrak{A}$) is a set of pairs:

$$\mathcal{A} = \{(T_1, n_1), \ldots, (T_m, n_m)\},$$

where $T_1, \ldots, T_m$ are distinct elements of $\mathfrak{A}$ and $n_i \geq 1$ for each $i = 1, \ldots, m$.

2. The number n_i is called the **multiplicity** of element T_i in the family $\mathcal{A}$.

3. We denote the length of the family $\mathcal{A}$ as
$$|\mathcal{A}| = n_1 + n_2 + \ldots + n_m$$

and $\mathcal{A}! = n_1! \cdot n_2! \cdot \ldots \cdot n_m!$

We use letters $\mathcal{A}, \Psi, \ldots$ for families. The same elements $T_1, \ldots, T_n \in \mathfrak{A}$ can be represented as a sequence or a family.

Definition 4.3

1. Any sequence $\alpha = (T_1, \ldots, T_n)$ **reduces** to a family:

$$\left\{\left(\overline{T}_1, n_1\right), \ldots, \left(\overline{T}_q, n_q\right)\right\},$$

where $\overline{T}_1, \ldots, \overline{T}_q$ are the elements $T_1, \ldots, T_n$ written without repetitions and each n_i is the number of times that $\overline{T}_i$ is repeated in α; $n_1 + \ldots + n_q = n$.

For each family $\mathcal{A}$ of length n there are $n!/\mathcal{A}!$ sequences that reduce to $\mathcal{A}$.

2. If a sequence α reduces to a family $\mathcal{A}$ we denote $\alpha! = \mathcal{A}!$

Lemma 4.1

Denote $M_k = \left\langle |X_{\overline{0}}|^k \right\rangle_0$ and

$$C(m, X) = \max\left\{M_{k_1} \cdot \ldots \cdot M_{k_l}\right\},$$

where the maximum is taken over all sequences of numbers $k_1, \ldots, k_l$ with $k_1 + \ldots + k_l = m$.

Denote $C_1 = 3e(4v + 1)$. For any sequence $\beta = (B_1, \ldots, B_n)$ of elements of R and any sequence $\tau = (t_1, \ldots, t_m)$ of elements of $\mathbb{Z}^v$, the following holds:

$$\left|\left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0\right| \leq C\left(|\tau|, X\right)(C_1)^{|\tau|+|\beta|} \tau! \, \beta! \, , \text{ where}$$

$$\left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0 = \left\langle X_{t_1}, \ldots, X_{t_m}, \Phi_{B_1}, \ldots, \Phi_{B_n} \right\rangle_0 , \quad \text{the semi-invariant with respect to } P_0.$$

Proof

Consider a sequence $\alpha = (A_1, \ldots, A_p)$ of elements of $\mathfrak{A}$ and the set $I = \{1, \ldots, p\}$ of their indices. Consider random variables $F_1, \ldots, F_p$ such that each F_i depends on X_t, $t \in A_i$ ($i = 1, \ldots, p$).

Denote $F_S = \prod_{i \in S} F_i$ for any $S \subseteq I$ and denote

$$D(\alpha) = \max\left|\left\langle F_{S_1} \right\rangle_0 \cdot \ldots \cdot \left\langle F_{S_l} \right\rangle_0\right|,$$

where the maximum is taken over all partitions $\{S_1, \ldots, S_l\}$ of I. If each random variable F_i ($i = 1, \ldots, p$) satisfies Carleman condition, then $D(\alpha) < \infty$.

Suppose the sequence α reduces to a family $\mathcal{A} = \left\{\left(\overline{A}_1, n_1\right), \ldots, \left(\overline{A}_q, n_q\right)\right\}$ and $J = J(\alpha)$. Theorem 1 on pg. 69 of (Malyshev and Minlos, 1991) implies that:

$$\left|\left\langle F_\alpha^\backslash \right\rangle_0\right| \leq \frac{3}{2} D(\alpha) \prod_{i \in I} (3v_i), \quad \text{where} \quad \left\langle F_\alpha^\backslash \right\rangle_0 = \left\langle F_1, \ldots, F_p \right\rangle_0.$$

Then

$$\left|\left\langle F_\alpha^\backslash \right\rangle_0\right| \leq \frac{3}{2} D(\alpha) 3^p \prod_{j \in J} \left(v_j\right)^{n_j} . \tag{10}$$

By the Estimation Theorem (Theorem 4.1) we have:

$$ln\left(\prod_{j \in J} \left(v_j\right)^{n_j}\right) = \sum_{j \in J} n_j \, ln \, v_j \leq |\alpha| \, ln \, L + \sum_{j \in J} n_j \, ln \, n_j ,$$

where $L = 4v + 1$. So

$$\prod_{j \in J} \left(v_j\right)^{n_j} \leq e^{|\alpha| \, ln \, L} \cdot e^{\sum_{j \in J} n_j \, ln \, n_j} = L^{|\alpha|} \prod_{j \in J} \left(n_j\right)^{n_j} .$$

From Stirling formula we have $k^k < \frac{1}{2} k! e^k$ for any positive integer k. Then by (10):

$$\left|\left\langle F_\alpha^\backslash \right\rangle_0\right| \leq \frac{3}{2} D(\alpha) 3^{|\alpha|} L^{|\alpha|} \prod_{j \in J} \left(n_j\right)^{n_j}$$

$$< D(\alpha)(3L)^{|\alpha|} \prod_{j \in J} n_j! \, e^{n_j} = D(\alpha)(3eL)^{|\alpha|} \alpha! \, .$$

Thus,

$$\left| \left\langle F_\alpha^\backslash \right\rangle_0 \right| \le D(\alpha)(C_1)^{|\alpha|} \alpha! \, . \tag{11}$$

Now we take $\alpha = \tau * \beta = (t_1,\ldots, t_m, B_1,\ldots, B_n)$, a concatenation of the sequences τ and β. We take $F_i = X_{t_i}$ for $i = 1,\ldots, m$ and $F_{m+j} = \Phi_{B_j}$ for $j = 1,\ldots, n$. Then by (11):

$$\left| \left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0 \right| \le D(\alpha)(C_1)^{|\tau|+|\beta|} \tau! \, \beta! \tag{12}$$

Since each $|\Phi_{B_j}| \le 1$, we have $D(\alpha) = D(\tau * \beta) \le$

$\max \left\{ \left\langle |X_{S_1}| \right\rangle_0 \cdot \ldots \cdot \left\langle |X_{S_l}| \right\rangle_0 \right\}$, where the maximum is

taken over all partitions $\{S_1,\ldots, S_l\}$ of $\tau = (t_1,\ldots, t_m)$. For a fixed partition $\{S_1,\ldots, S_l\}$ denote $k_i = |S_i|$, $i = 1,\ldots, l$; then $k_1 + \ldots + k_l = m$.

From Hölder inequality we get by induction for any random variables $Y_1,\ldots, Y_n$:

$$\left| \left\langle Y_1 \cdot \ldots \cdot Y_n \right\rangle_0 \right| \le \left(\left\langle |Y_1|^n \right\rangle_0 \right)^{\frac{1}{n}} \cdot \ldots \cdot \left(\left\langle |Y_n|^n \right\rangle_0 \right)^{\frac{1}{n}} \, .$$

So each $\left| \left\langle X_{S_i} \right\rangle_0 \right| \le \left\langle |X_{\bar{0}}|^{k_i} \right\rangle_0 = M_{k_i}$ because all X_t have

the same distribution. Therefore $\left| \left\langle X_{S_1} \right\rangle_0 \cdot \ldots \cdot \left\langle X_{S_l} \right\rangle_0 \right|$

$\le M_{k_1} \cdot \ldots \cdot M_{k_l}$ and $D(\alpha) \le C(m, X)$. Then by (12):

$$\left| \left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0 \right| \le C(m, X)(C_1)^{|\tau|+|\beta|} \tau! \, \beta! \, .$$

5. Proof of the Central Limit Theorem for the Interaction Model

5.1. Semi-Invariants with Respect to Gibbs Measure

In this subsection we prove a series of lemmas about estimates and semi-invariants and later we use these lemmas to prove the main theorem.

Definition 5.1

Suppose $\Psi = \{(B_1, n_1),\ldots, (B_k, n_k)\}$ is a family of elements of R.

1. We define its **associated graph** $G(\Psi)$ as follows.
For each $i = 1,\ldots, k$, B_i has the form $B_i = \{r_i, s_i\}$. The points r_i and s_i belong to the set of vertices of $G(\Psi)$ and

there are n_i edges between r_i and s_i. There are no other vertices or edges.

2. We say that the family Ψ **connects** a sequence τ of elements of $\mathbb{Z}^v$ if the associated graph $G(\Psi)$ is connected and the set of its vertices contains all elements of the sequence τ.

Thus, the associated graph has $|\Psi| = n_1 + \ldots + n_k$ edges. The mapping $\Psi \mapsto G(\Psi)$ is a one-to-one mapping of families of elements of R to this type of graphs on $\mathbb{Z}^v$.

A semi-invariant is a symmetrical functional, the order of random variables is not important. If a sequence β of elements of R reduces to a family Ψ, we denote $\left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0 = \left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0$.

Lemma 5.1

If a family Ψ of elements of R does not connect a sequence τ of elements of $\mathbb{Z}^v$, then $\left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0 = 0$.

Proof

Denote $G = G(\Psi)$. Fix a sequence β that reduces to the family Ψ. Suppose Ψ does not connect τ. There are two cases.

Case 1. Some elements of τ are not vertices of G.

Without loss of generality we can assume that exactly first q elements of τ are not vertices of G; $\tau = (t_1,\ldots, t_q, t_{q+1},\ldots, t_m)$ and $\beta = (B_1,\ldots, B_n)$. Then by Lemma 2.5.4), the random vectors $\left(X_{t_1},\ldots, X_{t_q} \right)$ and $\left(X_{t_{q+1}},\ldots, X_{t_m}, \Phi_{B_1},\ldots, \Phi_{B_n} \right)$ are independent of each other and by Lemma 2.1.2):

$$\left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0 = \left\langle X_{t_1},\ldots, X_{t_q}, X_{t_{q+1}},\ldots, X_{t_m}, \Phi_{B_1},\ldots, \Phi_{B_n} \right\rangle_0 = 0 \, .$$

Case 2. All elements of τ are vertices of G but G is not connected.

Then $G = G_1 \cup G_2$, where G_1 and G_2 are disjoint graphs. Without loss of generality we can assume: $\tau = (t_1,\ldots, t_q, s_1,\ldots, s_l)$ and $\beta = (A_1,\ldots, A_m, B_1,\ldots, B_n)$, where G_1 contains $t_1,\ldots, t_q$ and all elements of $A_1 \cup \ldots \cup A_m$ as vertices; G_2 contains $s_1,\ldots, s_l$ and all elements of $B_1 \cup \ldots \cup B_n$ as vertices. By Lemma 2.5.4), the random vectors $\left(X_{t_1},\ldots, X_{t_q}, \Phi_{A_1},\ldots, \Phi_{A_m} \right)$ and $\left(X_{s_1},\ldots, X_{s_l}, \Phi_{B_1},\ldots, \Phi_{B_n} \right)$ are independent of each other and by Lemma 2.1.2):

$$\left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0$$
$$= \left\langle X_{t_1},\ldots, X_{t_q}, \Phi_{A_1},\ldots, \Phi_{A_m}, X_{s_1},\ldots, X_{s_l}, \Phi_{B_1},\ldots, \Phi_{B_n} \right\rangle_0 = 0 \, .$$

The following lemma is mentioned by several authors without a proof or with a complicated proof. Here we provide a short, simple proof giving an explicit value for the estimation constant.

Lemma 5.2

Denote $C_2 = 4v^2$. Fix a sequence τ of points in $\mathbb{Z}^v$ and a natural number $n \geq 1$. The number of families Ψ such that $|\Psi| = n$ and Ψ connects τ, is not greater than $(C_2)^n$.

Proof

For a family $\Psi = \{(B_1, n_1),..., (B_k, n_k)\}$ consider the associated graph $G = G(\Psi)$. A new graph G' is obtained from G by adding for every edge another edge with the same ends. So G' has $2n$ edges. Each vertex of G' has an even degree and G' is connected, hence G' has an Eulerian trail, that is a closed path which includes every edge of the graph exactly once; the length of such a path is $2n$.

Therefore the number of the families with $|\Psi| = n$ that connect τ, is not greater than the number of paths with $2n$ steps through τ going along edges of the lattice $\mathbb{Z}^v$. There are at most $2v$ directions at each vertex. Therefore the number of such paths is not greater than $(2v)^{2n} = (C_2)^n$ for $C_2 = (2v)^2$.

For a sequence $\tau = (t_1,..., t_m)$ denote $\left\langle X_\tau^\backslash \right\rangle_{\lambda,N}$ $= \left\langle X_{t_1},...,X_{t_m} \right\rangle_{\lambda,N}$, the semi-invariant with respect to $P_{\lambda,N}$ and $\left\langle X_\tau^\backslash \right\rangle_\lambda = \left\langle X_{t_1},...,X_{t_m} \right\rangle_\lambda$, the semi-invariant with respect to P_λ. In the Definition 2.4 we expressed the measure $P_{\lambda,N}$ in terms of measure P_0. The following lemma describes a connection between semi-invariants with respect to these measures.

Lemma 5.3

Denote $C_3 = (2C_1C_2)^{-1}$, where C_1 is the constant from Lemma 4.1 and C_2 is the constant from Lemma 5.2. Fix $N > 1$ and a sequence τ of points in Λ_N. The following equality holds:

$$\left\langle X_\tau^\backslash \right\rangle_{\lambda,N} = \sum_{n=0}^\infty \lambda^n \sum_\Psi \frac{1}{\Psi!} \left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0, \tag{13}$$

where the finite inner sum is taken over all families $\Psi = \{(B_1, n_1),..., (B_k, n_k)\}$ such that $|\Psi| = n$, Ψ connects τ and each $B_i \subset \Lambda_N$. The series (13) converges absolutely and uniformly for $\lambda \in [-C_3, C_3]$.

Proof

The semi-invariant with respect to Gibbs measure can be expanded in Taylor series:

$$\left\langle X_\tau^\backslash \right\rangle_{\lambda,N} = \left\langle X_{t_1}, X_{t_2},...,X_{t_m} \right\rangle_{\lambda,N}$$
$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left\langle X_{t_1}, X_{t_2},...,X_{t_m}, \underbrace{U_N,...,U_N}_{n\ times} \right\rangle_0, \tag{14}$$

where $U_N = -\lambda \sum_{B \in R, B \subset \Lambda_N} \Phi_B$ is the potential of the interaction model.

The proof of (14) can be found in (Malyshev and Minlos, 1991), pg. 34. Expanding (14) we get:

$$\left\langle X_\tau^\backslash \right\rangle_{\lambda,N}$$
$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left\langle X_{t_1},...,X_{t_m}, -\lambda \sum_{B_1 \in R, B_1 \subset \Lambda_N} \Phi_{B_1},...,-\lambda \sum_{B_n \in R, B_n \subset \Lambda_N} \Phi_{B_n} \right\rangle_0$$
$$= \sum_{n=0}^\infty \frac{\lambda^n}{n!} \sum_{B_1,...,B_n \in R, each\ B_i \subset \Lambda_N} \left\langle X_{t_1},...,X_{t_m}, \Phi_{B_1},...,\Phi_{B_n} \right\rangle_0.$$

So

$$\left\langle X_\tau^\backslash \right\rangle_{\lambda,N} = \sum_{n=0}^\infty \lambda^n a_{N,n}, \tag{15}$$

where $a_{N,n} = \frac{1}{n!} \sum_\beta \left\langle X_\tau^\backslash, \Phi_\beta^\backslash \right\rangle_0$.

The last sum is finite and is taken over all sequences $\beta = (B_1, B_2,...,B_n)$ of elements of R such that each $B_i \subset \Lambda_N$. In this sum we can take only the sequences β that connect τ because for others the corresponding addends equal 0 by Lemma 5.1. If $n = 0$, then $a_{N,0} = \left\langle X_\tau^\backslash \right\rangle_0$.

For each family Ψ of length n there are $\dfrac{n!}{\Psi!}$ sequences that reduce to Ψ. Therefore

$$a_{N,n} = \sum_\Psi \frac{1}{\Psi!} \left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0, \tag{16}$$

where the sum is taken over all families $\Psi = \{(B_1, n_1),..., (B_k, n_k)\}$ such that $|\Psi| = n$, Ψ connects τ and each $B_i \subset \Lambda_N$. So we have proven the equality (13).

It remains to prove that the series converges absolutely and uniformly on $[-C_3, C_3]$. By Lemma 4.1:

$$|a_{N,n}| \leq \sum_\Psi \frac{1}{\Psi!} \left| \left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0 \right|$$
$$\leq \sum_\Psi \frac{1}{\Psi!} C(|\tau|, X)(C_1)^{|\tau|+n} \tau! \Psi! = \sum_\Psi (C_1)^n q(\tau),$$

where $q(\tau) = (C_1)^{|\tau|} \tau! \, C(|\tau|, X)$ does not depend on n. By Lemma 5.2, the number of addends in the sum (16) is not greater than $(C_2)^n$. So $|a_{N,n}| \le (C_2)^n (C_1)^n q(\tau)$ and for $\lambda \in [-C_3, C_3]$:

$$\left| \lambda^n a_{N,n} \right| \le \left(C_3 \right)^n \left(C_1 C_2 \right)^n q(\tau) = \frac{1}{2^n} q(\tau).$$

Therefore the series in (15) converges absolutely and uniformly on $[-C_3, C_3]$.

Lemma 5.4

Fix a sequence τ of points in $\mathbb{Z}^v$. For any $|\tau| < C_3$ (where C_3 is the constant from Lemma 5.3) the following equality holds:

$$\left\langle X_\tau^\backslash \right\rangle_\lambda = \sum_{n=0}^\infty \lambda^n \sum_\Psi \frac{1}{\Psi!} \left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0, \tag{17}$$

where the finite inner sum is taken over all families Ψ with $|\Psi| = n$ that connect τ. The series (17) converges absolutely and uniformly for $\lambda \in [-C_3, C_3]$.

Proof

It is proven in (Malyshev and Minlos, 1991) that $\lim_{N \to \infty} \left\langle X_\tau^\backslash \right\rangle_{\lambda, N} = \left\langle X_\tau^\backslash \right\rangle_\lambda$. By taking the limit of both sides of (13) as $N \to \infty$ we get the equality (17). Similarly to Lemma 5.3 we estimate the common term of the series (17), which proves its absolute and uniform convergence.

Lemma 5.5

If $|\lambda| < C_3$ (where C_3 is the constant from Lemma 5.3), then any finite set of the random variables X_t, $t \in \mathbb{Z}^v$, satisfy the Carleman condition with respect to measure P_λ.

Proof

Fix $t \in \mathbb{Z}^v$ and denote $\tau = (t)$, a sequence of length 1. Similarly to Lemma 5.4, we can show that for any natural l:

$$\left\langle X_t^{2l} \right\rangle_\lambda = \sum_{n=0}^\infty \lambda^n \sum_\Psi \frac{1}{\Psi!} \left\langle X_t^{2l}, \Phi_\Psi^\backslash \right\rangle_0, \tag{18}$$

where the finite inner sum is taken over all families Ψ with $|\Psi| = n$ that connect $\tau = (t)$.

Similarly to the proof of Lemma 5.3, we can estimate the coefficient for λ^n:

$$\left| \sum_\Psi \frac{1}{\Psi!} \left\langle X_t^{2l}, \Phi_\Psi^\backslash \right\rangle_0 \right| \le (C_2)^n \left\langle X_0^{2l} \right\rangle_0 (C_1)^{1+n},$$

since $|\tau| = \tau! = 1$. Then for $|\lambda| < C_3$:

$$\left\langle X_t^{2l} \right\rangle_\lambda \le \sum_{n=0}^\infty |\lambda|^n \left\langle X_0^{2l} \right\rangle_0 C_1 \left(C_1 C_2 \right)^n$$

$$\le \sum_{n=0}^\infty C_1 \left\langle X_0^{2l} \right\rangle_0 \left(C_3 C_1 C_2 \right)^n = C_1 \left\langle X_0^{2l} \right\rangle_0 \sum_{n=0}^\infty \frac{1}{2^n};$$

$$\left\langle X_t^{2l} \right\rangle_\lambda \le 2 C_1 \left\langle X_0^{2l} \right\rangle_0. \tag{19}$$

Since X_0 satisfies Carleman condition with respect to P_0 by conditions (4)-(5), then by (19) X_t satisfies Carleman condition with respect to P_λ. Since all X_t are identically distributed, then any finite set of them satisfy Carleman condition with respect to P_λ.

5.2. Estimation of Semi-Invariants for RG

In this subsection we estimate semi-invariants $\left\langle Y_{r_1}^{(k)}, Y_{r_2}^{(k)}, ..., Y_{r_m}^{(k)} \right\rangle_\lambda$, where each $Y_r^{(k)}$ is the result of RG transformation of the random field $\{X_t \mid t \in \mathbb{Z}^v\}$:

$$Y_r^{(k)} = k^{-\frac{\alpha}{2}} \sum_{t \in C_r^k} \left(X_t - \mu \right), \tag{20}$$

where $\langle X_t \rangle_\lambda = \mu$ does not depend on t.

Lemma 5.6

1. For any $r \in \mathbb{Z}^v$, $\left\langle Y_r^{(k)} \right\rangle_\lambda = 0$.

2. For $m > 1$, any $r_1, ..., r_m \in \mathbb{Z}^v$ and $|\lambda| < C_3$:

$$\left\langle Y_{r_1}^{(k)}, Y_{r_2}^{(k)}, ..., Y_{r_m}^{(k)} \right\rangle_\lambda = k^{-m\alpha/2} \sum_{n=0}^\infty \lambda^n \sum_\tau \sum_\Psi \frac{1}{\Psi!} \left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0, \tag{21}$$

where the first inner sum is over all sequences $\tau = (t_1, ..., t_m)$ with each $t_i \in C_{r_i}^k$ and the second inner sum is over all families Ψ with $|\Psi| = n$ that connect τ.

The series (21) converges absolutely and uniformly on $[-C_3, C_3]$.

Proof

1. Obvious

2. For $|\lambda| < C_3$ and any sequence $\tau = (t_1, ..., t_m)$ of elements of $\mathbb{Z}^v$ we have by Lemma 2.1.1) and 4):

$$\left\langle X_{t_1} - \mu, ..., X_{t_m} - \mu \right\rangle_\lambda = \left\langle X_{t_1}, ..., X_{t_m} \right\rangle_\lambda = \left\langle X_\tau^\backslash \right\rangle_\lambda.$$

So

$$\left\langle Y_{r_1}^{(k)}, ..., Y_{r_m}^{(k)} \right\rangle_\lambda = k^{\left(-\frac{\alpha}{2} \right)^m} \sum_\tau \left\langle X_\tau^\backslash \right\rangle_\lambda$$

$$= k^{-m\alpha/2} \sum_\tau \sum_{n=0}^\infty \lambda^n \sum_\Psi \frac{1}{\Psi!} \left\langle X_\tau^\backslash, \Phi_\Psi^\backslash \right\rangle_0$$

by Lemma 5.4, where the first sum is taken over all $\tau = (t_1, \ldots, t_m)$ with each $t_i \in C_{r_i}^k$ and the third sum is over all families Ψ with $|\Psi| = n$ that connect τ. By Lemma 5.4 for each τ the series $\sum_{n=0}^{\infty} \lambda^n \sum_{\Psi} \frac{1}{\Psi!} \left\langle X_\tau^{\backslash}, \Phi_\Psi^{\backslash} \right\rangle_0$ converges absolutely and uniformly on $[-C_3, C_3]$ and there are a finite number of τ in the sum, hence:

$$\left\langle Y_{r_1}^{(k)}, \ldots, Y_{r_m}^{(k)} \right\rangle_\lambda = k^{-m\alpha/2} \sum_{n=0}^{\infty} \lambda^n \sum_\tau \sum_\Psi \frac{1}{\Psi!} \left\langle X_\tau^{\backslash}, \Phi_\Psi^{\backslash} \right\rangle_0$$

and this series also converges absolutely and uniformly on $[-C_3, C_3]$.

Lemma 5.7

If $|\lambda| < C_3$, then each semi-invariant $\left\langle Y_{r_1}^{(k)}, \ldots, Y_{r_m}^{(k)} \right\rangle_\lambda$ is translation invariant.

Proof

If we shift a sequence $(r_1, r_2, \ldots, r_m)$ by vector r, then we get a new sequence $(r_1 + r, r_2 + r, \ldots, r_m + r)$. This shifts all sequences $\tau = (t_1, t_2, \ldots, t_m)$ and families Ψ by vector kr in (21) in Lemma 5.6. But it does not change the values of its addends because the field $\{X_t \mid t \in \mathbb{Z}^\nu\} \cup \{\Phi_B \mid B \in R\}$ is translation invariant (see Note 2 after Definition 2.3).

Lemma 5.8

For a fixed $t_1 \in \mathbb{Z}^\nu$ denote:

$$W(n, t_1, m) = \sum_{t_2, \ldots, t_m \in Z^\nu} \sum_\Psi \frac{1}{\Psi!} \left| \left\langle X_\tau^{\backslash}, \Phi_\Psi^{\backslash} \right\rangle_0 \right|,$$

where $\tau = (t_1, t_2, \ldots, t_m)$ and the second sum is taken over all families Ψ with $|\Psi| = n$ that connect τ. Then:

$$W(n, t_1, m) \le K(m)(n+1)^{m-1} \left(C_1 C_2 \right)^n, \tag{22}$$

where $K(m) = C(m, X)(C_1)^m m!$ and $C(m, X)$ is defined in Lemma 4.1.

Proof

For a non-zero addend in the sum, $|\Psi| = n$ and Ψ connects $(t_1, t_2, \ldots, t_m)$, hence each of $t_2, \ldots, t_m$ is a vertex in the associated graph $G(\Psi)$. So there are at most $n+1$ choices for each of them. By Lemma 5.2, there are at most $(C_2)^n$ families Ψ with $|\Psi| = n$ that connect τ. Using also Lemma 4.1, we get:

$$W(n, t_1, m) \le (n+1)^{m-1} (C_2)^n C(m, X)(C_1)^{m+n} m!$$

and

$$W(n, t_1, m) \le K(m)(n+1)^{m-1} \left(C_1 C_2 \right)^n.$$

Lemma 5.9

Suppose $|\lambda| < C_3$ and $r_1, \ldots, r_m \in \mathbb{Z}^\nu$. If $\alpha > \nu$ or ($\alpha = \nu$ and $m > 2$), then:

$$\lim_{k \to \infty} \left\langle Y_{r_1}^{(k)}, \ldots, Y_{r_m}^{(k)} \right\rangle_\lambda = 0.$$

Proof

Case $m = 1$ has been considered in Lemma 5.6.1). So we assume $m > 1$. To estimate the semi-invariant we use the series (21) from Lemma 5.6 and consider the corresponding series of absolute values:

$$A_k = k^{-m\alpha/2} \sum_{n=0}^{\infty} B_{n,k} |\lambda^n|,$$

where $B_{n,k} = \sum_{\tau, \Psi} \frac{1}{\Psi!} \left| \left\langle X_\tau^{\backslash}, \Phi_\Psi^{\backslash} \right\rangle_0 \right|$. Here the sum is taken over all sequences τ and families Ψ as in (21).

Then $\left| \left\langle Y_{r_1}^{(k)}, \ldots, Y_{r_m}^{(k)} \right\rangle_\lambda \right| \le A_k$ and it is sufficient to show that $\lim_{k \to \infty} A_k = 0$.

Using the notation from Lemma 5.8, we have:

$$B_{n,k} \le \sum_{t_1 \in C_r^k} W(n, t_1, m).$$

Here we omit the restriction that $t_i \in C_r^k$ for $i = 2, \ldots, m$. By Lemma 5.8:

$$W(n, t_1, m) \le K(m)(n+1)^{m-1} \left(C_1 C_2 \right)^n. \tag{23}$$

Since $t_1 \in C_r^k$ there are k^ν choices for t_1. Since $C_3 C_1 C_2 = \frac{1}{2}$, we have:

$$A_k \le k^{-m\alpha/2} \sum_{n=0}^{\infty} |\lambda|^n \sum_{t_1 \in C_r^k} W(n, t_1, m)$$

$$\le k^{-m\alpha/2} \sum_{n=0}^{\infty} (C_3)^n k^\nu K(m)(n+1)^{m-1} (C_1 C_2)^n$$

$$\le k^{-m\alpha/2} k^\nu K(m) \sum_{n=0}^{\infty} (n+1)^{m-1} \frac{1}{2^n}$$

$$\le k^{\nu - m\alpha/2} K(m) \frac{(m-1)!}{\left(1 - \frac{1}{2}\right)^m}$$

$$\le k^{\nu - m\alpha/2} K(m) 2^m (m-1)!$$

For the last inequality we used Lemma A.1.1) from Appendix.

If $\alpha > v$ or ($\alpha = v$ and $m > 2$), then $v - m\alpha/2 < 0$. So

$$0 \le \lim_{k \to \infty} A_k \le K(m) 2^m (m-1)! \lim_{k \to \infty} k^{v - m\alpha/2} = 0$$

and $\lim_{k \to \infty} A_k = 0$.

Corollary 5.1

If $\alpha > v$, then for any $r \in \mathbb{Z}^v$:

$$\lim_{k \to \infty} Var\left(Y_r^{(k)}\right) = 0 .$$

Proof

It follows from Lemma 5.9 because $Var\left(Y_r^{(k)}\right) = \left\langle Y_r^{(k)}, Y_r^{(k)} \right\rangle_\lambda$.

5.3. Finding the limiting covariances

From here till the end of this section we consider only the case when $\alpha = v$ and $m = 2$. Other cases are investigated earlier.

Lemma 5.10

Suppose $|\lambda| < C_3$. If $r_1, r_2 \in \mathbb{Z}^v$ and $r_1 \ne r_2$, then

$$\lim_{k \to \infty} \left\langle Y_{r_1}^{(k)}, Y_{r_2}^{(k)} \right\rangle_\lambda = 0 . \tag{24}$$

Proof

We consider four cases.

Case 1: $r_1 = \bar{0}$ and the first coordinate of r_2 is negative.

Denote $r = r_2$. Clearly, for any $t = (t_1,..., t_v) \in C_{\bar 0}^k$, we have: $0 \le t_1 \le k-1$. Similarly, for any $s \in C_r^k$, $s_1 \le -1$. We introduce cross-sections of the cube $C_{\bar 0}^k$:

$$D_l = \left\{ t = (t_1,..., t_v) \in C_{\bar 0}^k \mid t_1 = l \right\}, l = 0, 1,..., k-1 .$$

Clearly, $C_{\bar 0}^k = \bigcup_{l=0}^{k-1} D_l$. Next we show:

if $t \in D_l$, $s \in C_r^k$ and Ψ connects (t, s), then $|\Psi| \ge l+1$. (25)

For $t \in D_l$ we have $t_1 = 1$. For $s \in C_r^k$ we have $s_1 \le -1$. So the distance between such t and s is at least $l + 1$.

If a family Ψ connects (t, s), then $|\Psi| \ge l + 1$. This proves (25).

By Lemma 5.4:

$$k^v \left\langle Y_{\bar 0}^{(k)}, Y_r^{(k)} \right\rangle_\lambda = \sum_{t \in C_{\bar 0}^k} \sum_{s \in C_r^k} \left\langle X_t, X_s \right\rangle_\lambda$$

$$= \sum_{t \in C_{\bar 0}^k} \sum_{s \in C_r^k} \sum_{n=0}^\infty \sum_\Psi \frac{\lambda^n}{\Psi!} \left\langle X_t, X_s, \Phi_\Psi^\backslash \right\rangle_0 , \tag{26}$$

where the last sum is taken over all families Ψ with $|\Psi| = n$ that connect (t, s).

By (25) and since the series converges absolutely we have:

$$k^v \left\langle Y_{\bar 0}^{(k)}, Y_r^{(k)} \right\rangle_\lambda = \sum_{l=0}^{k-1} \sum_{t \in D_l} \sum_{s \in C_r^k} \sum_{n=0}^\infty \sum_\Psi \frac{\lambda^n}{\Psi!} \left\langle X_t, X_s, \Phi_\Psi^\backslash \right\rangle_0$$

$$= \sum_{l=0}^{k-1} \sum_{t \in D_l} \sum_{n=l+1}^\infty \sum_\Psi \sum_{s \in C_r^k} \frac{\lambda^n}{\Psi!} \left\langle X_t, X_s, \Phi_\Psi^\backslash \right\rangle_0 ,$$

where $|\Psi| = n$ and Ψ connects (t, s).

For $\tau = (t, s)$ we have: $|\tau| = 2$, $t \ne s$ and $\tau! = 1$. Then by Lemma 4.1 for $m = 2$ we get:

$$\left| \left\langle X_t, X_s, \Phi_\Psi^\backslash \right\rangle_0 \right| \le C(2, X)(C_1)^{n+2} \Psi!$$

Assume Ψ and $t \in D_l$ are fixed, $|\Psi| = n$ and Ψ connects (t, s). Then the graph $G(\Psi)$ has at most $n + 1$ vertices. At least $l + 1$ of them have nonnegative first coordinates. In order for Ψ to connect (t, s), the point s should be among the vertices of $G(\Psi)$ with negative first coordinates. Therefore there are at most $n + 1 - (l + 1) = n - l$ choices for s. So

$$\sum_\Psi \sum_{s \in C_r^k} \left| \frac{\lambda^n}{\Psi!} \left\langle X_t, X_s, \Phi_\Psi^\backslash \right\rangle_0 \right|$$

$$\le (C_2)^n (n-l) \frac{1}{\Psi!} C(2, X)(C_1)^{n+2} \Psi! |\lambda|^n$$

$$\le C(2, X)(C_1)^2 (n-l) |\lambda C_1 C_2|^n \le K(2)(n-l) \frac{1}{2^n} ,$$

where $K(2) = 2C(2, X)(C_1)^2$ as in Lemma 5.8.

$$\sum_{n=l+1}^\infty (n-l) \frac{1}{2^n} = \frac{1}{2^{l+1}} \sum_{n=l+1}^\infty (n-l) \frac{1}{2^{n-l-1}}$$

$$= \frac{1}{2^{l+1}} \sum_{k=1}^\infty k \frac{1}{2^{k-1}} = \frac{1}{2^{l+1}} \frac{1}{\left(1 - \frac{1}{2}\right)^2} = \frac{1}{2^{l-1}}$$

Since D_l contains k^{v-1} points, we have:

$$\left|\left\langle Y_{\bar 0}^{(k)}, Y_r^{(k)}\right\rangle_\lambda\right| \le k^{-\nu} K(2) \sum_{l=0}^{k-1} k^{\nu-1} \sum_{n=l+1}^{\infty} (n-l) \frac{1}{2^n}$$

$$= \frac{K(2)}{k} \sum_{l=0}^{k-1} \frac{1}{2^{l-1}} \le \frac{K(2)}{k} \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{2^l}$$

$$= \frac{K(2)}{2k} \cdot 2 = \frac{K(2)}{k}$$

Therefore $\qquad \left|\left\langle Y_{\bar 0}^{(k)}, Y_r^{(k)}\right\rangle_\lambda\right| \le \frac{K(2)}{k} \qquad$ and

$\lim_{k\to\infty} \left\langle Y_{\bar 0}^{(k)}, Y_r^{(k)}\right\rangle_\lambda = 0$.

Case 2: The first coordinate of r_1 is greater than the first coordinate of r_2. Then by Lemma 5.7, $\left\langle Y_{r_1}^{(k)}, Y_{r_2}^{(k)}\right\rangle_\lambda = \left\langle Y_{\bar 0}^{(k)}, Y_{r_2-r_1}^{(k)}\right\rangle_\lambda$. Thus, Case 2 is reduced to Case 1.

Case 3: The first coordinate of r_1 is less than the first coordinate of r_2. This is reduced to Case 2 by interchanging r_1 and r_2.

Case 4: The general case. Since $r_1 \ne r_2$, they should differ in at least one coordinate, for example, in j-th coordinate. The proof is obtained by applying the proofs in Cases 1-3 to j-th coordinates instead of the first coordinates.

5.4. Finding the Limiting Variances

In the following theorem we derive explicit expressions for the limiting variances of $Y_r^{(k)}$. This theorem is interesting by itself and also becomes a part of the direct proof of the main theorem.

Theorem 5.1

Suppose $|\lambda| < C$, where $C = \min\left\{ C_3, \frac{\sigma^2}{8K(2)C_1 C_2} \right\}$

and σ^2 is the variance of $X_{\bar 0}$ with respect to P_0. Then for any $r \in \mathbb{Z}^\nu$:

$$\lim_{y\to\infty} \left\langle Y_r^{(k)}, Y_r^{(k)}\right\rangle_\lambda = \sum_{n=0}^{\infty} \lambda^n V_n = \sigma^2 + \sum_{n=1}^{\infty} \lambda^n V_n > 0 .$$

Here the series converges absolutely and uniformly for $\lambda \in [-C, C]$ and:

$$V_n = \sum_{t\in Z^\nu} \sum_\Psi \frac{1}{\Psi!} \left\langle X_{\bar 0}, X_t, \Phi_\Psi^{\backslash}\right\rangle_0, \tag{27}$$

where the second sum is taken over all families Ψ with $|\Psi| = n$ that connect $(\bar 0, t)$. For $n = 0$, $V_0 = \left\langle X_{\bar 0}, X_{\bar 0}\right\rangle_0 = \sigma^2$.

Proof

Suppose $|\lambda| < C$. For any $s \in C_{\bar 0}^k$ denote:

$$\tilde W(n,s,k) = \sum_{t\in C_{\bar 0}^k} \sum_\Psi \frac{1}{\Psi!} \left\langle X_s, X_t, \Phi_\Psi^{\backslash}\right\rangle_0 . \tag{28}$$

Here the second sum is taken over all families Ψ with $|\Psi| = n$ that connect (s, t).

If $n = 0$, then in (28) we have $\Psi = \varnothing$, $\left\langle X_s, X_t, \Phi_\Psi^{\backslash}\right\rangle_0 = \langle X_s, X_t\rangle_0$. For $t \ne s$, $\langle X_s, X_t\rangle_0 = 0$. So $\tilde W(0,s,k) = \langle X_s, X_s\rangle_0 = \sigma^2$. Thus,

$$\tilde W(0,s,k) = \sigma^2 . \tag{29}$$

Using Lemma 5.8 for $m = 2$, we get:

$$\left|V_n\right| \le K(2)(n+1)(C_1 C_2)^n; \tag{30}$$

$$\left|\tilde W(n,s,k)\right| \le K(2)(n+1)(C_1 C_2)^n . \tag{31}$$

For $n \ge 0$ and $k > 2n$ consider a cube $K_n^k \subset C_{\bar 0}^k$:

$$K_n^k = \{s = (s_1,\ldots,s_\nu) \in \mathbb{Z}^\nu \mid n \le s_i < k-n \text{ for each } i=1,\ldots,\nu\}.$$

Next we prove:

If $s \in K_n^k$, then $\tilde W(n,s,k) = V_n$. $\tag{32}$

Proof of (32)

Suppose $s \in K_n^k$. We will show that the sums (27) and (28) for V_n and $\tilde W(n,s,k)$, respectively, contain equal addends.

Consider an addend $\left\langle X_{\bar 0}, X_t, \Phi_\Psi^{\backslash}\right\rangle_0$ in (27); here $|\Psi| = n$ and Ψ connects $(\bar 0, t)$. Translating all points by s, we get $\left\langle X_s, X_{t+s}, \Phi_{\Psi+\bar s}^{\backslash}\right\rangle_0 = \left\langle X_{\bar 0}, X_t, \Phi_\Psi^{\backslash}\right\rangle_0$ and $B + \bar s$ connects $(s, t + s)$. Since $\|t - \bar 0\| \le |\Psi| = n$ and $s \in K_n^k$, we get $t + s \in C_{\bar 0}^k$. Thus, for every addend in (27) there is an equal addend in (28). Similarly we can show that for every addend in (28) there is an equal addend in (27). The proof of (32) is completed.

Next we estimate the variance of $Y_r^{(k)}$, which equals $\left\langle Y_r^{(k)}, Y_r^{(k)}\right\rangle_\lambda$. By Lemma 5.7 and Lemma 5.6 we have:

$$\left\langle Y_r^{(k)}, Y_r^{(k)} \right\rangle_\lambda = \left\langle Y_0^{(k)}, Y_0^{(k)} \right\rangle_\lambda$$

$$= k^{-\nu} \sum_{n=0}^\infty \lambda^n \sum_{s,t \in C_0^k} \sum_\Psi \frac{1}{\Psi!} \left\langle X_t, X_s, \Phi_\Psi^\backslash \right\rangle_0 \qquad (33)$$

$$= \sum_{n=0}^\infty \lambda^n k^{-\nu} \sum_{s \in C_0^k} \tilde{W}(n,s,k).$$

By (31) and because C_0^k contains k^ν points, we have:

$$\left| \lambda^n k^{-\nu} \sum_{s \in C_0^k} \tilde{W}(n,s,k) \right| \le |\lambda|^n k^{-\nu} \sum_{s \in C_0^k} \left| \tilde{W}(n,s,k) \right|$$

$$\le |\lambda|^n k^{-\nu} k^\nu K(2)(n+1)(C_1 C_2)^n \le K(2)(n+1)\frac{1}{2^n}$$

Therefore the series (33) converges absolutely and uniformly for any k. We split the inner sum $\sum_{s \in C_0^k} \tilde{W}(n,s,k)$ in two sums:

$$U_1(n,k) = \sum_{s \in K_n^k} \tilde{W}(n,s,k)$$

and

$$U_2(n,k) = \sum_{s \in C_0^k \backslash K_n^k} \tilde{W}(n,s,k).$$

So

$$\left\langle Y_r^{(k)}, Y_r^{(k)} \right\rangle_\lambda$$
$$= \sum_{n=0}^\infty \lambda^n \left[k^{-\nu} U_1(n,k) + k^{-\nu} U_2(n,k) \right] \qquad (34)$$

Since K_n^k contains $(k-2n)^\nu$ points, then by (32):

$$U_1(n,k) = \sum_{s \in K_n^k} \tilde{W}(n,s,k) = (k-2n)^\nu V_n$$

and

$$\lim_{k \to \infty} k^{-\nu} U_1(n,k) = \lim_{k \to \infty} \frac{(k-2n)^\nu}{k^\nu} V_n = V_n.$$

The difference of cubes $C_0^k \backslash K_n^k$ contains $k^\nu - (k-2n)^\nu$ points, so by (31) we have:

$$|U_2(n,k)| \le \sum_{s \in C_0^k \backslash K_n^k} |\tilde{W}(n,s,k)|$$

$$\le K(2)(n+1)(C_1 C_2)^n \left[k^\nu - (k-2n)^\nu \right]$$

and

$$0 \le \lim_{k \to \infty} |k^{-\nu} U_2(n,k)|$$

$$\le K(2)(n+1)(C_1 C_2)^n \lim_{k \to \infty} \frac{k^\nu - (k-2n)^\nu}{k^\nu} = 0$$

By (34) and since the series converges uniformly for any k:

$$\lim_{k \to \infty} \left\langle Y_r^{(k)}, Y_r^{(k)} \right\rangle_\lambda = \sum_{n=0}^\infty \lambda^n \left[\lim_{k \to \infty} k^{-\nu} U_1(n,k) + \lim_{k \to \infty} k^{-\nu} U_2(n,k) \right]$$

$$= \sum_{n=0}^\infty \lambda^n V_n = \sigma^2 + \sum_{n=1}^\infty \lambda^n V_n$$

because $V_0 = \sigma^2$.

It remains to show that the limit is positive. By (30) and Lemma A.1.2) from Appendix:

$$\sum_{n=1}^\infty |\lambda^n V_n| \le \sum_{n=1}^\infty |\lambda^n| K(2)(n+1)(C_1 C_2)^n$$

$$= K(2) \sum_{n=1}^\infty (n+1) |\lambda C_1 C_2|^n$$

$$\le K(2) \frac{2|\lambda C_1 C_2|}{(1-|\lambda C_1 C_2|)^2} \le 8K(2) C_1 C_2 |\lambda| < \sigma^2$$

$$because\ |\lambda C_1 C_2| < \frac{1}{2}\ and\ |\lambda| < \frac{\sigma^2}{8K(2) C_1 C_2}$$

Therefore

$$\sum_{n=1}^\infty \lambda^n V_n > -\sigma^2 \text{ and } \sigma^2 + \sum_{n=1}^\infty \lambda^n V_n > 0.$$

Proof of the Main Theorem (Theorem 3.1)

Suppose $|\lambda| < C$, where C is the constant from Theorem 5.1.

1. *Case $\alpha > \nu$.*

By Lemma 5.6.1) the limiting expectation of each $Y_r^{(k)}$ is 0 and by Corollary 5.1 the limiting variance is 0. This proves part 1 of the theorem.

2. *Case $\alpha = \nu$.*

By Lemma 5.6.1) and Lemma 5.9 for $m \ne 2$:

$$\lim_{k \to \infty} \left\langle Y_{r_1}^{(k)}, \dots, Y_{r_m}^{(k)} \right\rangle_\lambda = 0.$$

By Lemma 5.10 the limiting covariances equal 0 and by Theorem 5.1 the limiting variance is positive.

Thus, all of the limiting semi-invariants equal 0, except the variance. Therefore the random variables $Y_{r_1}^{(k)}, Y_{r_2}^{(k)}, ..., Y_{r_m}^{(k)}$ converge in distribution as $k \to \infty$ to an independent multivariate normal random vector, due to Lemma 2.4. The statement about the variance of the limiting distribution follows from Theorem 5.1.

6. Conclusion

In this paper we introduce a concept of interaction model and prove a generalization of the central limit theorem to a random field in this model transformed by renormalization group. We show that as $k \to \infty$ the resulting random fields $Y_r^{(k)}$ converge in distribution to an independent random field with Gaussian distribution. We find the limits of all semi-invariants of $Y_r^{(k)}$ as $k \to \infty$ and apply Carleman theorem. In particular, we show that all the semi-invariants, except the variances, tend to 0.

In Theorem 5.1 we give an explicit expression for the limiting variance. In order to find the limiting semi-invariants, we derive estimations of the semi-invariants of the original random field with respect to Gibbs measure.

We provide a more transparent proof under more general conditions for the inequality about the number of links in a set with a symmetric binary relation (Theorem 4.1). In this theorem and the lemmas about estimations of semi-invariants, as well as in the main theorem, we derive explicit expressions for the estimation constants.

A possible direction for future research is generalization of our theorem to other models in statistical mechanics.

Appendix A

Lemma A.1

If $0 \leq x < 0.5$, then the following series converges absolutely for any $l = 1, 2, ...$ and

1) $\sum_{n=0}^{\infty} (n+1)^l x^n \leq \dfrac{l!}{(1-x)^{l+1}}$; \hfill (35)

2) $\sum_{n=0}^{\infty} (n+1) x^n \leq \dfrac{2x}{(1-x)^2}$.

Proof

1) We have

$$(n+1)^l x^n \leq (n+1)(n+2)...(n+l) x^n = \left(x^{n+l}\right)^{(l)}.$$

Next:

$$\sum_{n=0}^{\infty} \left(x^{n+l}\right)^{(l)} = \left(\sum_{n=0}^{\infty} x^n\right)^{(l)} = \left(\frac{1}{1-x}\right)^{(l)}$$

because for $n < l$ we have $(x^n)^{(l)} = 0$. It is easily proven by induction on l that:

$$\left(\frac{1}{1-x}\right)^{(l)} = \frac{l!}{(1-x)^{l+1}}.$$

So the series (35) converges and satisfies the inequality.

2) If $l = 1$, then

$$\sum_{n=0}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2}$$

and

$$\sum_{n=1}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2} - 1 \leq \frac{2x}{(1-x)^2}. \quad \square$$

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Author's Contributions

Each author contributed to the design of the research, the main results, proofs, and writing of the manuscript.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues are involved.

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