Journal of Mathematics and Statistics

Original Research Paper

New Bivariate Exponentiated Modified Weibull Distribution

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Article history
Received: 05-08-2017
Revised: 19-09-2017
Accepted: 31-10-2017

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Abstract: The new modified Weibull distribution named exponentiated modified Weibull distribution introduced by many authors. In this study, we define and study a bivariate modified Weibull distribution. Some joint probability functions and conditional probability density function are obtained. Also, joint survival function is expressed in compact form. Several statistical properties of this distribution are discussed. In the area of stress-strength models, we estimate the reliability follow a bivariate exponentiated modified Weibull distribution with dependence between stress and strength variables. Parameters estimators using the maximum likelihood method are obtained. Moreover, a numerical illustration is used to obtain Maximum Likelihood Estimators (MLEs).

Keywords: Generalized Modified Weibull Distribution, Maximum Likelihood Estimators, Bivariate Distributions, Stress-Strength Models

Introduction

Sarhan and Zaindin (2009) introduced the new class from the Weibull distribution named modified Weibull distribution. The random variable Y has univariate Modified Weibull (MW) distribution with parameters a, b and c if its Cumulative Distribution Function (CDF) has the following form:

$$G_{MW}(y;a,b,c) = 1-\exp\{-(ay+by^c)\}, y \geq 0 \quad (1)$$

where, a, b, $c > 0$.

The Modified Weibull Distribution (MWD), generalizes several well- known distributions. Among of these distributions are the exponential distribution, the linear failure rate distribution, the Rayleigh distribution and the Weibull distribution.

The Exponentiated Modified Weibull Distribution (EMWD) introduced in Elbatal (2011). The random variable Y has univariate EMWD with parameters a, b, c and v if its CDF is given as:

$$G_{EMW}(y;a,b,c,v) = \left(1-e^{-(ay+by^c)}\right)^v, y \geq 0 \quad (2)$$

where, v, a, b, $c > 0$.

This paper introduces a Bivariate Exponentiated Modified Weibull Distribution (BEMWD) by using the method of Marshall and Olkin (1986). Several bivariate distributions are derived by using this method, Marshall and Olkin (1986). Several bivariate exponential distribution introduced in Sarhan and Balakrishnan (2007). A new class of bivariate Gompertz distribution is introduced in Al-Khedhairi and El-Gohary (2011). The bivariate generalized exponential distribution proposed by Kundu and Gupta (2009), a new bivariate generalized Gompertz distribution presented in El-Sherpieny *et al.* (2013) and Marshall-Olkin bivariate Weibull distribution studied in Kundu and Gupta (2013); Mustafa (2016) introduced a new bivariate distribution with generalized quadratic hazard rate marginals.

The BMWD generalizes several distributions as following:

- The bivariate generalized exponential distribution, BGED (v, a), when $b \to 0$
- The bivariate exponentiated linear failure rate distribution, BELFRD (v, a, b), when $c = 2$
- The bivariate exponentiated Rayleigh distribution, BERD (v, b), when $a \to 0$, $c = 2$
- The bivariate exponentiated Weibull distribution, BEWD (v, a, b), when $a \to 0$

The rest of the paper is organized as follows. Some properties of the BGQHRD are presented in Section 2. Section 3, presents the reliability analysis. Section 4, gives the stress-strength model. The parameter estimations using maximum likelihood is given in Section 5. We use a set of real data in Section 6 as an application. Conclusions for the article is introduced in Section 7.

A New Bivariate Exponentiated Modified Weibull Distribution

In this section, we discuss the new Bivariate Exponentiated Modified Weibull Distribution (BEMWD). We start with the joint cumulative distribution function of the distribution.

Suppose V_1, V_2 and V_3 are three independent random variables such that $V_i \sim EMW(v_i, a, b, c)$ for $i = 1,2,3$.

Define:

$$Y_1 = \max(V_1, V_3), Y_2 = \max(V_2, V_3)$$

Theorem 1

The joint cumulative distribution function of Y_1, Y_2 is:

$$G_{Y_1 Y_2}(y_1, y_2) = P[Y_1 \le y_1, Y_2 \le y_2] = \prod_{i=1}^{3} G_{EMW}(y_i; v_i, a, b, c) \quad (3)$$

where, $y_3 = \min(y_1, y_2)$

Proof

The joint cumulative distribution function is defined as:

$$\begin{aligned} G(y_1, y_2) &= P[Y_1 \le y_1, Y_2 \le y_2] \\ &= P[\max(V_1, V_3) \le y_1, \max(V_2, V_3) \le y_2] \\ &= P[V_1 \le y_1, V_2 \le y_2, V_3 \le \min(y_1, y_2)] \end{aligned}$$

But the random variables V_1, V_2 and V_3 are mutually independent, then:

$$\begin{aligned} G(y_1, y_2) &= P[V_1 \le y_1] P[V_2 \le y_2] P[V_3 \le \min(y_1, y_2)] \\ &= \prod_{i=1}^{3} G_{EMW}(y_i; v_i, a, b, c) \end{aligned} \quad (4)$$

This completes the proof.

The joint PDF of the BEMW can be obtained as the following theorem.

Theorem 2

If $(Y_1, Y_2) \sim BEMW(v_1, v_2, v_3, a, b, c)$, then the joint PDF of (Y_1, Y_2) is give by:

$$g_{Y_1, Y_2}(y_1, y_2) = \begin{cases} g_1(y_1, y_2) & if \quad y_1 < y_2 \\ g_2(y_1, y_2) & if \quad y_2 < y_1 \\ g_3(y, y) & if \quad y_1 = y_2 = y \end{cases} \quad (5)$$

Where:

$$g_1(y_1, y_2) = g_{EMW}(y_2; v_2, a, b, c) g_{EMW}(y_1; v_1 + v_3, a, b, c) \quad (6)$$

$$g_2(y_1, y_2) = g_{EMW}(y_1; v_1, a, b, c) g_{EMW}(y_2; v_2 + v_3, a, b, c) \quad (7)$$

$$g_3(y, y) = \left(\frac{v_3}{v_1 + v_2 + v_3} \right) g_{EMW}(y; v_1 + v_2 + v_3, a, b, c) \quad (8)$$

Proof

We can derive the expression for $g_1(y_1,y_2)$ and $g_2(y_1,y_2)$ by differentiating the joint CDF given in Equation (3) with respect to y_1 and y_2. But can be derived by using the following identity:

$$\int_0^\infty \int_0^{y_2} g_1(y_1, y_2) dy_1 dy_2 + \int_0^\infty \int_0^{y_1} g_2(y_1, y_2) dy_2 dy_1 + \int_0^\infty g_3(y, y) dy = 1$$

Then, we can find mathematically that:

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^{y_2} g_1(y_1, y_2) dy_1 dy_2 \\ &= \int_0^\infty v_2 (a + bcy_2^{c-1}) e^{-(ay_2 + by_2^c)} \left[1 - e^{-(ay_2 + by_2^c)} \right]^{v_1 + v_2 + v_3 - 1} dy_2. \end{aligned} \quad (9)$$

Similarly:

$$\begin{aligned} I_2 &= \int_0^\infty \int_0^{y_1} g_2(y_1, y_2) dy_2 dy_1 \\ &= \int_0^\infty v_1 (a + bcy_1^{c-1}) e^{-(ay_1 + by_1^c)} \left[1 - e^{-(ay_1 + by_1^c)} \right]^{v_1 + v_2 + v_3 - 1} dy_1. \end{aligned} \quad (10)$$

From Equations (9) and (10), we then get:

$$\begin{aligned} \int_0^\infty g_3(y, y) dy &= 1 - I_1 - I_2 \\ &= \int_0^\infty v.(a + bcy^{c-1}) e^{-(ay + by^c)} \left[1 - e^{-(ay + by^c)} \right]^{v.-1} dx \\ &- \int_0^\infty v_2 (a + bcy^{c-1}) e^{-(ay + by^c)} \left[1 - e^{-(ay + by^c)} \right]^{v.-1} dy \\ &- \int_0^\infty v_1 (a + bcy^{c-1}) e^{-(ay + by^c)} \left[1 - e^{-(ay + by^c)} \right]^{v.-1} dy \\ &= \int_0^\infty v_3 (a + bcy^{c-1}) e^{-(ay + by^c)} \left[1 - e^{-(ay + by^c)} \right]^{v.-1} dy. \end{aligned}$$

where, $v. = v_1+v_2+v_3$.

That is:

$$g_3(y, y) = v_3 (a + bcx^{c-1}) e^{-(ay + by^c)} \left[1 - e^{-(ay + by^c)} \right]^{v_1 + v_2 + v_3 - 1} \quad (11)$$

The proof is completed.

The marginal density function of Y_i, $i = 1,2$ can be obtained as follows.

Theorem 3

The marginal PDF of Y_i $i = 1,2$ is given by:

$$g_{Y_i}(y_i) = g_{EMW}(y_i; v_i + v_3, a, b, c) \quad (12)$$

where, v_i, a, b, $c > 0$, $i = 1,2$.

Proof

The marginal CDF of Y_i $i = 1,2$, say $F_{Y_i}(y_i)$, is:

$$G_{Y_i}(y_i) = P\left[Y_i \le y_i\right] = P\left[\max(V_i, V_3) \le y_i\right] = P\left[V_i \le y_i, V_3 \le y_i\right]$$

Since V_i and V_3 are independent, we get:

$$G_{Y_i}(y_i) = P\left[Y_i \le y_i\right]P\left[V_3 \le y_i\right] = \left[1 - e^{-(ay_i + by_i^c)}\right]^{v_i + v_3} = G_{EMW}(y_i; v_i + v_3, a, b, c) \quad (13)$$

The formula given in Equation (12) can be obtained by differentiating Equation (13) with respect to y_i.

That is, $Y_i \sim EMWD(v_i + v_3, a, b, c)$.

The following theorem gives the rth moments of Y_i, $i = 1,2$.

Theorem 4

The rth moment of $Y_i (i = 1,2)$ derived by:

$$\mu'_r = E\left[Y_i^r\right] = (v_i + v_3)\sum_{j=0}^{v_i+v_3-1}\sum_{k=0}^{\infty}\binom{v_i + v_3 - 1}{j}\frac{(-1)^{j+k}\beta^k}{k!a^{kc+r+1}} (j+1)^{(1-c)k-r-1}\left[a\Gamma(kc + r + 1) + \frac{bc\Gamma((k+1)\gamma + r)}{[(j+1)a]^{c-1}}\right]. \quad (14)$$

Proof

Starting with:

$$\mu'_r = E\left[Y_i^r\right] = \int_0^\infty y_i^r g_{Y_i}(y_i)dy_i$$

and from Equation (12), we get:

$$E\left[Y_i^r\right] = (v_i + v_3)\int_0^\infty y_i^r (a + bcy_i^{c-1})e^{-(ay_i + by_i^c)} \left[1 - e^{-(ay_i + by_i^c)}\right]^{v_i + v_3 - 1} dy_i. \quad (15)$$

Since $0 < e^{-(ay_i + by_i^c)} < 1$, for $y>0$ then using the binomial expansion of $\left[1 - e^{-(ay_i + by_i^c)}\right]^{v_i + v_3 - 1}$ given by:

$$\left[1 - e^{-(ay_i + by_i^c)}\right]^{v_i + v_3 - 1} = \sum_{j=0}^{v_i+v_3-1}\binom{v_i + v_3 - 1}{j}(-1)^j e^{-j(ay_i + by_i^c)}. \quad (16)$$

Substituting from Equation (16) into (15), we get:

$$E[Y_i^r] = (v_i + v_3)\sum_{j=0}^{v_i+v_3-1}\binom{v_i + v_3 - 1}{j}(-1)^j \times \int_0^\infty y_i^r (a + bcy_i^{c-1})e^{-(j+1)(ay_i + by_i^c)}dy_i.$$

Using the series expansion of $e^{-(j+1)by_i^c}$, one gets:

$$E[Y_i^r] = (v_i + v_3)\sum_{j=0}^{v_i+v_3-1}\sum_{k=0}^{\infty}\binom{v_i + v_3 - 1}{j}\frac{(-1)^{j+k}[(j+1)b]^k}{k!} \int_0^\infty (a + bcy_i^{c-1})y_i^{kc+r}e^{-(j+1)ay_i}dy_i$$

$$= (v_i + v_3)\sum_{j=0}^{v_i+v_3-1}\sum_{k=0}^{\infty}\binom{v_i + v_3 - 1}{j}\frac{(-1)^{j+k}[(j+1)b]^k}{k!} \int_0^\infty \left[ay_i^{kc+r} + bcy_i^{(k+1)c+r-1}\right]e^{-(j+1)ay_i}dy_i$$

Let $v = (j+1)ay_i$, in the above integral, then we can get:

$$E[Y_i^r] = (v_i + v_3)\sum_{j=0}^{v_i+v_3-1}\sum_{k=0}^{\infty}\binom{v_i + v_3 - 1}{j}\frac{(-1)^{j+k}[(j+1)b]^k}{k![(j+1)a]^{kc+r+1}} \left[a\Gamma(kc + r + 1) + \frac{bc\Gamma((k+1)c + r)}{[(j+1)a]^{c-1}}\right]$$

Thus (14) is obtained.

The following theorem gives the conditional PDF.

Theorem 5

The conditional probability density function of Y_i, give $Y_j = y_j$ is given by:

$$g_{Y_i|Y_j}(y_i|y_j) = \begin{cases} g^{(1)}_{Y_i|Y_j}(y_i|y_j) & \text{if } y_i < y_j \\ g^{(2)}_{Y_i|Y_j}(y_i|y_j) & \text{if } y_j < y_i \\ g^{(3)}_{Y_i|Y_j}(y_i|y_j) & \text{if } y_i < y_j \end{cases} \quad (17)$$

where:

$$g^{(1)}_{Y_iY_j}(y_i|y_j) = \left(\frac{v_j}{v_j + v_3}\right)\frac{g_{Y_i}(y_i, v_i + v_3, a, b, c)}{G_{Y_j}(y_j; v_3, a, b, c)} = \frac{1}{(v_j + v_3)\left[1 - e^{-(ay_j + by_j^c)}\right]^{v_3}}[v_j(v_i + v_3) (a + bcy_i^{c-1})e^{-(ay_i + by_i^c)}\left[1 - e^{-(ay_i + by_i^c)}\right]^{v_i + v_3 - 1}] \quad (18)$$

$$g^{(2)}_{Y_iY_j}(y_i|y_j) = g_{Y_i}(y_i; v_i, a, b, c) = v_i(a + bcy_i^{c-1}) - e^{-(ay_i + by_i^c)}\left[-e^{-(ay_i + by_i^c)}\right]^{v_i - 1} \quad (19)$$

$$g^{(3)}_{Y_i|Y_j}(y_i|y_j) = \left(\frac{v_3}{v_j+v_3}\right) G_{Y_i}(y_i; v_i, a, b, c) = \left(\frac{v_3}{v_j+v_3}\right)\left[1-e^{-(ay_i+by_i^c)}\right]^{v_i}. \quad (20)$$

Proof

Since the conditional PDF is given by:

$$g_{Y_i|Y_j}(y_i|y_j) = \frac{g_{Y_iY_j}(y_i y_j)}{g_{Y_j}(y_j)}$$

The theorem can be proved by substituting form Equations (5) and (12), in the above relation.

Reliability Analysis

The joint survival function of (Y_1, Y_2), CDF of the random variables $Z = \max\{Y_1,Y_2\}$ and $W = \min\{Y_1,Y_2\}$ can be derived in this section. The random variable $Z = \max\{Y_1,Y_2\}$ represents the lifetime for a parallel system with two components and $W = \min\{Y_1,Y_2\}$ is the lifetime of a series system with two components.

Theorem 6

The joint survival function of (Y_1, Y_2) is obtained as:

$$S_{Y_1,Y_2}(y_1,y_2) = \begin{cases} S_1(y_1,y_2) & \text{if } y_1 < y_2 \\ S_2(y_1,y_2) & \text{if } y_2 < y_1 \\ S_0(y_1,y_2) & \text{if } y_1 = y_2 = y \end{cases} \quad (21)$$

where:

$$S_1(y_1,y_2) = 1-\left[1-e^{-(ay_2+by_2^c)}\right]^{v_2+v_3} - \left[1-e^{-(ay_1+by_1^c)}\right]^{v_1+v_3} \times \left(1-\left[1-e^{-(ay_2+bx_2^c)}\right]^{v_2}\right),$$

$$S_2(y_1,y_2) = 1-\left[1-e^{-(ay_1+by_1^c)}\right]^{v_1+v_2} - \left[1-e^{-(ay_2+by_2^c)}\right]^{v_2+v_3} \times \left(1-\left[1-e^{-(ay_1+bx_1^c)}\right]^{v_1}\right),$$

$$S_0(y,y) = 1-\left[1-e^{-(ay+by^c)}\right]^{v_3}\left(1-\left[1-e^{-(ay+by^c)}\right]^{v_1} + \left[1-e^{-(ay+by^c)}\right]^{v_2}\left[1-e^{-(ay+by^c)}\right]^{v_1+v_2}\right).$$

Proof

One can obtain Equation (21) by using the relation:

$$S_{Y_1Y_2}(y_1,y_2) = 1-G_{Y_1}(y_1)-G_{Y_2}(y_2)+G_{Y_1Y_2}(y_1,y_2) \quad (22)$$

Substituting from Equations (3) and (13) into Equation (22), we can obtain Equation (21). This completes the proof.

Also, the bivariate failure rate function can be obtained by substituting from Equations (5) and (21) in the following relation, Basu (1971); Johnson and Kotz (1975):

$$h_{Y_1,Y_2}(y_1,y_2) = \frac{g_{Y_1,Y_2}(y_1,y_2)}{S_{Y_1,Y_2}(y_1,y_2)}. \quad (23)$$

Lemma 1.

The random variable $Z = \max\{Y_1, Y_2\}$ following CDF:

$$G_Z(z) = \left[1-e^{-(az+bz^c)}\right]^{v_1+v_2+v_3} \quad (24)$$

Proof

Since:

$$G_Z(z) = P[Z \le z] = P[\max\{Y_1,Y_2\} \le z]$$
$$= P[Y_1 \le z, Y_2 \le z]$$
$$= P[\max\{V_1,V_3\} \le z, \max\{V_2,V_3\} \le z]$$
$$= P[V_1, \le z, V_2 \le z, V_3 \le z]$$

But V_1, V_2 and V_3 are independent random variables, so:

$$G_Z(z) = P[V_1 \le z]P[V_2 \le z]P[V_3 \le z] = \prod_{i=1}^{3} G_{EMW}(z; v_i, a, b, c) \quad (25)$$

From Equation (2) into (25), we get (24).

We note that, $Z \sim EMWD(v_1 + v_2 + v_3, a, b, c)$.

Lemma 2

Let $W = \min\{Y_1, Y_2\}$ be the random variable, the $G_W(w)$ can be derived as:

$$G_W(w) = \left[1-e^{-(aw+bw^c)}\right]^{v_1} + \left[1-e^{-(aw+bw^c)}\right]^{v_2} - \left[1-e^{-(aw+bw^c)}\right]^{v_1+v_2+v_3} \quad (26)$$

Proof

Since:

$$G_W(w) = P[W \le w] = P[\min\{Y_1,Y_2\} \le w]$$
$$= 1-P[\min\{Y_1,Y_2\} > w] \quad (27)$$
$$= 1-P[Y_1 > w, Y_2 > w] = 1-S(w,w).$$

Substituting from Equation (22) into Equation (27), we get:

$$G_w(w) = G_{Y_1}(w) + G_{Y_2}(w) - G_{Y_1,Y_2}(w,w) \tag{28}$$

We can obtain Equation (26) by substituting from Equations (3) and (13) into Equation (28).

Stress-Strength Model

In this section, we consider the problem of the stress-strength model. The reliability $R = P[Y_1 < Y_2]$ which defined as a life of a component which has a random variable strength Y_2 and is subjected to random stress Y_1. We derive the form of R when (Y_1, Y_2) follow a BEM (ν_1, ν_2, ν_3, a, b, c) with dependence between Y_1 and Y_2.

Theorem 7.

For the random variables (Y_1, Y_2) with BEMW, then $R = P(Y_1 < Y_2)$ is:

$$R = \nu_2(\nu_1+\nu_3)\sum_{i=0}^{\nu_2-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\nu_1+\nu_3-1}\sum_{\ell=0}^{\infty}\binom{\nu_2-1}{i}\times \binom{\nu_1+\nu_3-1}{k}\frac{(-1)^{i+j+k+\ell}(k+1)^{\ell}b^{j+\ell}}{j!\ell!a^{jc+1}(i+1)^{j(c-1)+1}}\times \left[a(jc)!\sum_{m=0}^{jc}\psi_m + \frac{bc[(j+1)c-1]!}{[(i+1)a]^{c-1}}\sum_{m=0}^{(j+1)c-1}\psi_m\right], \tag{29}$$

where:

$$\psi_m = \frac{[(i+1)a]^m}{m![(k+i+2)a]^{m+\ell c+1}}\times \left[a\Gamma(m+\ell c+1) + \frac{bc\Gamma(m+(\ell+1)c)}{[(k+i+2)a]^{c-1}}\right].$$

Proof

Since:

$$R = P(Y_1 < Y_2) = \int_0^\infty\int_{y_1}^\infty g(y_1,y_2)dy_2dy_1 \tag{30}$$

Substituting from Equation (6) into Equation (30), we get:

$$R = \int_0^\infty\int_{y_1}^\infty \nu_2(a+bcy_2^{c-1})e^{-(ay_2+by_2^c)}\left[1-e^{-(ay_2+by_2^c)}\right]^{\nu_2-1}\times (\nu_1+\nu_3)(a+bcy_2^{c-1})e^{-(ay_1+by_1^c)}\left[1-e^{-(ay_1+by_1^c)}\right]^{\nu_1+\nu_3-1}dy_2dy_1.$$

By using the binomial expansion for $\left[1-e^{-(ay_2+by_2^c)}\right]^{\nu_2-1}$, we get:

$$R = \sum_{i=0}^{\nu_2-1}\binom{\nu_2-1}{i}(-1)^i\nu_2(\nu_1+\nu_3)\int_0^\infty\int_{y_1}^\infty(a+bcy_2^{c-1})e^{-(i+1)(ay_2+by_2^c)} (a+bcy_1^{c-1})e^{-(ay_1+by_1^c)}\left[1-e^{-(ay_1+by_1^c)}\right]^{\nu_1+\nu_3-1}dy_2dy_1$$

Using the series expansion for $e^{-(i+1)by_2^c}$, we have:

$$R = \nu_2(\nu_1+\nu_3)\sum_{i=0}^{\nu_2-1}\sum_{j=0}^{\infty}\binom{\nu_2-1}{i}(-1)^{i+j}\frac{[(i+1)b]^j}{j!}\times \int_0^\infty(a+bcy_2^{c-1})e^{-(ay_1+by_1^c)}\left[1-e^{-(ay_1+by_1^c)}\right]^{\nu_1+\nu_3-1}\times \int_{y_1}^\infty(ay_2^{cj}+bcy_2^{c(j+1)-1})e^{-(i+1)ay_2}dy_2dy_1 \tag{31}$$

Let $u = (i+1)ay_2$, then from Equation (31), we get:

$$R = \nu_2(\nu_1+\nu_3)\sum_{i=0}^{\nu_2-1}\sum_{j=0}^{\infty}\binom{\nu_2-1}{i}\frac{(-1)^{i+j}}{j!}[(i+1)b]^j \int_0^\infty(a+bcy_1^{c-1})e^{-(ay_1+by_1^c)}\left[1-e^{-(ay_1+by_1^c)}\right]^{\nu_1+\nu_3-1} \left[\frac{a\Gamma(cj+1,(i+1)ay_1)}{[(i+1)a]^{jc+1}} + \frac{bc\Gamma(c(j+1),(i+1)ay_1)}{[(i+1)a]^{(j+1)c}}\right]dy_1 \tag{32}$$

where, $\Gamma(s, x)$ is the incomplete Gamma function:

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt \tag{33}$$

Since $0 < e^{-(ay_1+by_1^c)} < 1$, by using binomial expansion:

$$\left[1-e^{-(ay_1+by_1^c)}\right]^{\nu_1+\nu_3-1} = \sum_{k=0}^{\nu_1+\nu_3-1}\binom{\nu_1+\nu_3-1}{k}(-1)^k e^{-k(ay_1+by_1^c)}. \tag{34}$$

Substituting from Equation (34) into Equation (32), we get:

$$R = \nu_2(\nu_1+\nu_3)\sum_{i=0}^{\nu_2-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\nu_1+\nu_3-1}\binom{\nu_2-1}{i}\binom{\nu_1+\nu_3-1}{k}(-1)^{i+j+k}\times \frac{[(i+1)b]^j}{j!(i+1)a]^{jc+1}}\int_0^\infty(a+bcy_1^{c-1})e^{-(k+1)(ay_1+by_1^c)}\times \left[a\Gamma(cj+1,(i+1)ay_1) + \frac{bc\Gamma(c(j+1),(i+1)ay_1)}{[(i+1)a]^{c-1}}\right]dy_1.$$

Since $0 < e^{-(k+1)by_1^c} < 1$, by using the series expansion, then:

$$R = \nu_2(\nu_1+\nu_3)\times\sum_{i=0}^{\nu_2-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\nu_1+\nu_3-1}\sum_{\ell=0}^{\infty}\binom{\nu_2-1}{i}\binom{\nu_1+\nu_3-1}{k}\times \frac{(-1)^{i+j+k+\ell}(k+1)^{\ell}b^{j+\ell}}{j!\ell!a^{jc+1}(i+1)^{j(c-1)+1}}\left[aI_1 + \frac{bc}{[(i+1)a]^{c-1}}I_2\right] \tag{35}$$

where:

$$I_1 = \int_0^\infty(a+bcy_1^{c-1})y_1^{\ell c}e^{-(k+1)ay_1}\Gamma(jc+1,(i+1)ay_1)dy_1, \tag{36}$$

$$I_2 = \int_0^\infty (a + bcy_1^{c-1}) y_1^{\ell c} e^{-(k+1)ay_1} \Gamma((j+1)c, (i+1)ay_1) dy_1. \quad (37)$$

If c is an integer then by using the following identity:

$$\Gamma((n,z) = (n-1)! e^{-z} \sum_{m=0}^{n-1} \frac{z^m}{m!}. \quad (38)$$

One can evaluate I_1 and I_2 as follows:

$$I_1 = (jc)! \sum_{m=0}^{jc} \frac{[(i+1)a]^m}{m![(k+i+2)a]^{m+\ell c+1}} \times \left[a\Gamma(m+\ell c+1) + \frac{bc\Gamma(m+(\ell+1)c)}{[(k+i+2)a]^{c-1}} \right] = (jc)! \sum_{m=0}^{jc} \psi_m, \quad (39)$$

where:

$$\psi_m = \frac{[(i+1)a]^m}{m![(k+i+2)a]^{m+\ell c+1}} \times \left[a\Gamma(m+\ell c+1) + \frac{bc\Gamma(m+(\ell+1)c)}{[(k+i+2)a]^{c-1}} \right]$$

and

$$I_2 = ((j+1)c-1)! \sum_{m=0}^{(j+1)c-1} \psi_m \quad (40)$$

Substituting from Equations (39) and (40) into Equation (35), we get:

$$R = \nu_2(\nu_1+\nu_3) \sum_{i=0}^{\nu_2-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\nu_1+\nu_3-1} \sum_{\ell=0}^{\infty} \binom{\nu_2-1}{i} \binom{\nu_1+\nu_3-1}{k} \times \frac{(-1)^{i+j+k+\ell}(k+1)^\ell b^{j+\ell}}{j!\ell! a^{jc+1}(i+1)^{j(c-1)+1}} \times \left[a(jc)! \sum_{m=0}^{jc} \psi_m + \frac{bc[(j+1)c-1]}{[(i+1)_a]^{c-1}} \sum_{m=0}^{(j+1)c-1} \psi_m \right] \quad (41)$$

This completes the proof.

Maximum Likelihood Estimators

We consider constant values to the parameters a = 0.012 and b = 2.159×10^{-8}, (Sarhan and Zaindin, 2009). We want to estimate the unknown parameters v_1, v_2, v_3 and c of the BEMWD. The maximum likelihood method can be used to estimate these unknown parameters.

Let $((y_{11}, y_{21}))$, (y_{12}, y_{22}),…, (y_{1m}, y_{2m})) be a random sample from BEMW(v_1, v_2, v_3, a, b, c), suppose the following notation:

$$m_1 = (i; y_{1i} < y_{2i}), \qquad m_2 = (i; y_{1i} > y_{2i})$$
$$m_3 = (i; y_{1i} = y_{2i} = y_i), \quad m = m_1 + m_2 + m_3.$$

For the sample of size m, the likelihood function is given by:

$$L(\Phi) = \prod_{i=1}^{m_1} g_1(y_{1i}, y_{2i}) \prod_{i=1}^{m_2} g_2(y_{1i}, y_{2i}) \prod_{i=1}^{m_3} g_3(y_i, y_i), \quad (42)$$

where, $\Phi = (v_1, v_2, v_3, a, b, c)$.

Substituting from Equations (6,7) and (8) into Equation (42), we get:

$$L(\Phi) = \prod_{i=1}^{m_1} \Big\{ \nu_2 (a + bcy_{2}^{c-1}) e^{-(ay_{2i}+by_{2i}^c)} [1 - e^{-(ay_{2i}+by_{2i}^c)}]^{\nu_2-1} (\nu_1+\nu_3)(a+bcy_{1i}^{c-1}) e^{-(ay_{1i}+by_{1i}^c)} \left[1 - e^{-(ay_{1i}+by_{1i}^c)}\right]^{\nu_1+\nu_3-1} \Big\}$$
$$\prod_{i=1}^{m_2} \Big\{ \nu_1 (a + bcy_{1i}^{c-1}) e^{-(ay_{1i}+by_{1i}^c)} \left[1 - e^{-(ay_{1i}+by_{1i}^c)}\right]^{\nu_1-1} (\nu_2+\nu_3) (a+bcy_{2i}^{c-1}) e^{-(ay_{2i}+by_{2i}^c)} \left[1 - e^{-(ay_{2i}+by_{2i}^c)}\right]^{\nu_2+\nu_3-1} \Big\}$$
$$\prod_{i=1}^{m_3} \Big\{ \nu_3 (a + bcy_{i}^{c-1}) e^{-(ay_i+by_i^c)} \left[1 - e^{-(ay_i+by_i^c)}\right]^{\nu_1+\nu_2+\nu_3-1} \Big\}$$

The log-likelihood function can be obtained as:

$$\begin{aligned}\mathcal{L}(\Phi) &= m_1 \ln(\nu_2) + \prod_{i=1}^{m_1} \ln(a+bcy_{2i}^{c-1}) - \sum_{i=1}^{m_1}(ay_{2i}+by_{2i}^c) \\ &+(\nu_2-1)\sum_{i=1}^{m_1} \ln\left[1-e^{-(ay_{2i}+by_{2i}^c)}\right] + m_1 \ln(\nu_1+\nu_3) + \\ &\sum_{i=1}^{m_1} \ln(a+bcy_{1i}^{c-1}) - \sum_{i=1}^{m_1}(ay_{1i}+by_{1i}^c) + (\nu_1+\nu_3-1)\times \\ &\sum_{i=1}^{m_1} \ln\left[1-e^{-(ay_{1i}+by_{1i}^c)}\right] + m_2 \ln(\nu_1) + \sum_{i=1}^{m_2} \ln(a+bcy_{1i}^{c-1}) - \\ &\sum_{i=1}^{m_2}(ay_{1i}+by_{1i}^c) + (\nu_1-1)\sum_{i=1}^{m_2} \ln\left[1-e^{-(ay_{1i}+by_{1i}^c)}\right] + \\ &m_2 \ln(\nu_2+\nu_3) + \sum_{i=1}^{m_2} \ln(a+bcy_{2i}^{c-1}) - \sum_{i=1}^{m_2}(ay_{2i}+by_{2i}^c) \\ &+(\nu_2+\nu_3-1)\sum_{i=1}^{m_2} \ln\left[1-e^{-(ay_{2i}+by_{2i}^c)}\right] + m_3 \ln(\nu_3) + \\ &\sum_{i=1}^{m_3} \ln(a+bcy_{1}^{c-1}) - \sum_{i=1}^{m_3}(ay_i+by_i^c) + (\nu_1+\nu_2+\nu_3-1) \\ &\sum_{i=1}^{m_3} \ln\left[1-e^{-(ay_i+by_i^c)}\right].\end{aligned} \quad (43)$$

The first partial derivatives of Eq. (43) with respect to v_1, v_2, v_3 and c are given as in the following forms:

$$\frac{\partial \mathcal{L}}{\partial \nu_1} = \frac{m_1}{\nu_1+\nu_3} + \sum_{i=1}^{m_1} \ln\left[1-e^{-(ay_{1i}+by_{1i}^c)}\right] + \frac{m_2}{\nu_1} + \sum_{i=1}^{m_2} \ln\left[1-e^{-(ay_{1i}+by_{1i}^c)}\right] + \sum_{i=1}^{m_3} \ln\left[1-e^{-(ay_{1i}+by_{1i}^c)}\right] \quad (44)$$

$$\frac{\partial \mathcal{L}}{\partial \nu_2} = \frac{m_1}{\nu_2} + \sum_{i=1}^{m_1} \ln\left[1 - e^{-(ay_{2i}+by_{2i}^c)}\right] + \frac{m_2}{\nu_2+\nu_3} + \sum_{i=1}^{m_2} \ln\left[1 - e^{-(ay_{2i}+by_{2i}^c)}\right] + \sum_{i=1}^{m_3} \ln\left[1 - e^{-(ay_i+by_i^c)}\right], \quad (45)$$

$$\frac{\partial \mathcal{L}}{\partial \nu_3} = \frac{m_1}{\nu_1+\nu_3} + \sum_{i=1}^{m_1} \ln\left[1 - e^{-(ay_{1i}+by_{1i}^c)}\right] + \frac{m_2}{\nu_2+\nu_3} + \sum_{i=1}^{m_2} \ln\left[1 - e^{-(ay_{2i}+by_{2i}^c)}\right] + \frac{m_3}{\nu_3} + \sum_{i=1}^{m_3} \ln\left[1 - e^{-(ay_i+by_i^c)}\right]. \quad (46)$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c} = b\Bigg[&\sum_{i=1}^{m_1}\mathcal{F}(y_{2i}) + (\nu_2-1)\sum_{i=1}^{m_1} y_{2i}^c \ln(y_{2i})\mathcal{A}(y_{2i}) - \\ &\sum_{i=1}^{m_2} y_{2i}^c \ln(y_{2i}) + \sum_{i=1}^{m_1}\mathcal{F}(y_{1i}) + (\nu_1+\nu_3-1)\times \\ &\sum_{i=1}^{m_1} y_{1i}^c \ln(y_{1i})\mathcal{A}(y_{1i}) - \sum_{i=1}^{m_1} y_{1i}^c \ln(y_{1i}) + \sum_{i=1}^{m_2}\mathcal{F}(y_{1i}) - \\ &\sum_{i=1}^{m_2} y_{1i}^c \ln(y_{1i}) + (\nu_1-1)\sum_{i=1}^{m_2} y_{1i}^c \ln(y_{1i})\mathcal{A}(y_{1i}) + \\ &\sum_{i=1}^{m_2}\mathcal{F}(y_{2i}) - \sum_{i=1}^{m_2} y_{2i}^c \ln(y_{2i}) + (\nu_2+\nu_3-1)\times \\ &\sum_{i=1}^{m_2} y_{2i}^c \ln(y_{2i})\mathcal{A}(y_{2i}) + \sum_{i=1}^{m_3}\mathcal{F}(y_i) - \sum_{i=1}^{m_3} y_i^c \ln(y_i) \\ &+(\nu_1+\nu_2+\nu_3-1)\sum_{i=1}^{m_3} y_i^c \ln(y_i)\mathcal{A}(y_i)\Bigg],\end{aligned} \quad (47)$$

where:

$$\mathcal{F}(y_i) = \frac{y_i^{c-1}[1+c\ \ln(y_i)]}{a+bcy_i^{c-1}}, \quad \mathcal{A}(y_i) = \frac{e^{-(ay_i+by_i^c)}}{1-e^{-(ay_i+by_i^c)}}.$$

Setting the first partial derivatives equal to zeros and solve the system of nonlinear equations with respect to v_1, v_2, v_3 and c to get the MLEs. The Equations (44) to (47) are not easy to solve, so numerical technique is needed to get the MLEs.

Based on the asymptotic distributions of the MLEs, the approximate confidence intervals of the parameters v_1, v_2, v_3 and c are derived. The variance-covariance matrix may be approximated by:

$$S = I^{-1} \quad (48)$$

where, I is the information matrix as follows:

$$I = E\begin{bmatrix} -\frac{\partial^2\mathcal{L}}{\partial\nu_1^2} & -\frac{\partial^2\mathcal{L}}{\partial\nu_1\partial\nu_2} & -\frac{\partial^2\mathcal{L}}{\partial\nu_1\partial\nu_3} & -\frac{\partial^2\mathcal{L}}{\partial\nu_1\partial c} \\ -\frac{\partial^2\mathcal{L}}{\partial\nu_2\partial\nu_1} & -\frac{\partial^2\mathcal{L}}{\partial\nu_2^2} & -\frac{\partial^2\mathcal{L}}{\partial\nu_2\partial\nu_3} & -\frac{\partial^2\mathcal{L}}{\partial\nu_2\partial c} \\ -\frac{\partial^2\mathcal{L}}{\partial\nu_3\partial\nu_1} & -\frac{\partial^2\mathcal{L}}{\partial\nu_3\partial\nu_2} & -\frac{\partial^2\mathcal{L}}{\partial\nu_3^2} & -\frac{\partial^2\mathcal{L}}{\partial\nu_3\partial c} \\ -\frac{\partial^2\mathcal{L}}{\partial c\partial\nu_1} & -\frac{\partial^2\mathcal{L}}{\partial c\partial\nu_2} & -\frac{\partial^2\mathcal{L}}{\partial c\partial\nu_3} & -\frac{\partial^2\mathcal{L}}{\partial c^2} \end{bmatrix}$$

The second partial derivatives can be obtained as follows:

$$\frac{\partial^2\mathcal{L}}{\partial\nu_1^2} = -\frac{m_1}{(\nu_1+\nu_3)^2} - \frac{m_2}{\nu_1^2}, \quad \frac{\partial^2\mathcal{L}}{\partial\nu_1\partial\nu_2} = 0,$$

$$\frac{\partial^2\mathcal{L}}{\partial\nu_1\partial\nu_3} = -\frac{m_1}{(\nu_1+\nu_3)^2}, \quad \frac{\partial^2\mathcal{L}}{\partial\nu_2^2} = -\frac{m_1}{\nu_2^2} - \frac{m_2}{(\nu_2+\nu_3)^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\nu_2\partial\nu_3} = -\frac{m_2}{(\nu_2+\nu_3)^2}, \quad \frac{\partial^2\mathcal{L}}{\partial\nu_3^2} = -\frac{m_1}{(\nu_1+\nu_3)^2} - \frac{m_2}{(\nu_2+\nu_3)^2} - \frac{m_3}{\nu_3^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial c\partial\nu_1} = b\left[\sum_{i=1}^{m_1} y_{1i}^c \ln(y_{1i})\mathcal{A}(y_{1i}) + \sum_{i=1}^{m_2} y_{1i}^c \ln(y_{1i})\mathcal{A}(y_{1i}) + \sum_{i=1}^{m_3} y_i^c \ln(y_i)\mathcal{A}(y_i)\right],$$

$$\frac{\partial^2\mathcal{L}}{\partial c\partial\nu_2} = b\left[\sum_{i=1}^{m_1} y_{2i}^c \ln(y_{2i})\mathcal{A}(y_{2i}) + \sum_{i=1}^{m_2} y_{2i}^c \ln(y_{2i})\mathcal{A}(y_{2i}) + \sum_{i=1}^{m_3} y_i^c \ln(y_i)\mathcal{A}(y_i)\right],$$

$$\frac{\partial^2\mathcal{L}}{\partial c\partial\nu_3} = b\left[\sum_{i=1}^{m_1} y_{1i}^c \ln(y_{1i})\mathcal{A}(y_{1i}) + \sum_{i=1}^{m_2} y_{2i}^c \ln(y_{2i})\mathcal{A}(y_{2i}) + \sum_{i=1}^{m_3} y_i^c \ln(y_i)\mathcal{A}(y_i)\right],$$

$$\begin{aligned}\frac{\partial^2 L}{\partial c^2} = &\sum_{i=1}^{m_1}\left(\begin{array}{c}\frac{2bx_{2i}^{c-1}\ln(y_{2i}) + bcy_{2i}^{c-1}(\ln(y_{2i}))^2}{a+bcy_{2i}^{c-1}} \\ -\frac{(by_{2i}^{c-1}+bcy_{2i}^{c-1}\ln(y_{2i}))^2}{(a+bcy_{2i}^{c-1})^2}\end{array}\right) + (\nu_2-1)\times \\ &\sum_{i=1}^{m_1}\left(by_{2i}^c(\ln(y_{2i}))^2\mathcal{A}(y_{2i}) - b^2y_{2i}^{2c}(\ln(y_{2i}))^2[A(y_{2i})+\mathcal{A}^2(y_{2i})]\right) \\ &+\sum_{i=1}^{m_1}\left(\begin{array}{c}\frac{2by_{1i}^{c-1}\ln(y_{1i}) + bcy_{1i}^{c-1}(\ln(y_{1i}))^2}{a+bcy_{1i}^{c-1}} \\ -\frac{by_{1i}^{c-1}+bcy_{1i}^{c-1}\ln(y_{1i}))^2}{(a+bcy_{1i}^{c-1})^2}\end{array}\right) + (\nu_1+\nu_3-1)\times \\ &\sum_{i=1}^{m_1}\left(by_{1i}^c(\ln(y_{1i}))^2A(y_{1i}) - b^2y_{1i}^{2c}(\ln(y_{1i}))^2\right. \\ &\left.\times[A(y_{1i})+A^2(y_{1i})]\right) - \sum_{i=1}^{m_1} by_{2i}^c(y_{2i}))^2 - \\ &\sum_{i=1}^{m_1} by_{2i}^c(\ln(y_{1i}))^2 - \sum_{i=1}^{m_2} by_{1i}^c(\ln(y_{1i}))^2 + \\ &\sum_{i=1}^{m_2}\left(\begin{array}{c}\frac{2by_{1i}^{c-1}\ln(y_{1i}) + bcy_{1i}^{c-1}(\ln(y_{1i}))^2}{a+bcy_{1i}^{c-1}} \\ -\frac{(by_{1i}^{c-1}+bcy_{1i}^{c-1}(\ln(y_{1i}))^2}{(a+bcy_{1i}^{c-1})^2}\end{array}\right) + (\nu_1-1)\times \\ &\sum_{i=1}^{m_2}\left(by_{1i}^c(\ln(y_{1i}))^2\mathcal{A}(y_{1i}) - b^2y_{1i}^{2c}(\ln(y_{1i}))^2\times[\mathcal{A}(y_{1i})+\mathcal{A}^2(y_{1i})]\right) + \\ &\sum_{i=1}^{m_2}\left(\begin{array}{c}\frac{2by_{2i}^{c-1}\ln(y_{2i}) + bcy_{2i}^{c-1}(\ln(y_{2i}))^2}{a+bcy_{2i}^{c-1}} \\ -\frac{(by_{2i}^{c-1}+bcy_{2i}^{c-1}(\ln(y_{2i}))^2}{(a+bcy_{2i}^{c-1})^2}\end{array}\right) + (\nu_2+\nu_3-1)\times \\ &\sum_{i=1}^{m_3}\left(by_{2i}^c(\ln(y_{2i}))^2\mathcal{A}(y_{2i}) - b^2y_{2i}^{2c}(\ln(y_{2i}))^2\times[\mathcal{A}(y_{2i})+\mathcal{A}^2(y_{2i})]\right) - \\ &\sum_{i=1}^{m_2} by_{2i}^c(\ln(y_{2i}))^2 - \sum_{i=1}^{m_3} by_i^c(\ln(y_i))^2 + (\nu_1+\nu_2+\nu_3-1)\times \\ &\sum_{i=1}^{m_3}\left(by_i^c(\ln(y_i))^2\mathcal{A}(y_i) - b^2y_i^{2c}(\ln(y_i))^2\times[\mathcal{A}(y_i)+\mathcal{A}^2(y_i)]\right) \\ &+\sum_{i=1}^{m_3}\left(\frac{2by_i^{c-1}\ln(y_i) + bcy_i^{c-1}(\ln(y_i))^2}{a+bcy_i^{c-1}} - \frac{(by_i^{c-1}+bcy_i^{c-1}(\ln(y_i))^2}{(a+bcy_i^{c-1})^2}\right).\end{aligned}$$

By using Equation (48), we can compute the (1-δ)100% confidence interval for v_1, v_2, v_3 and c respectively as:

Table 1. American football league data

Y_1	2.05	9.05	0.85	3.43	7.78	10.57	7.05
Y_2	3.98	9.05	0.85	3.43	7.78	14.28	7.05
Y_1	2.58	7.23	6.85	32.45	5.78	13.80	7.25
Y_2	2.58	9.68	34.58	42.35	25.98	49.75	7.25
Y_1	4.25	1.65	6.42	4.22	15.53	2.90	7.02
Y_2	4.25	1.65	15.08	9.48	15.53	2.90	7.02
Y_1	6.42	10.4	0.75	3.88	0.75	11.63	1.38
Y_2	6.42	10.25	2.98	6.43	0.75	17.37	1.38
Y_1	10.53	12.13	14.58	11.82	5.52	31.13	14.58
Y_2	10.53	12.13	14.58	11.82	11.27	49.88	20.57
Y_1	10.15	8.87	17.83	10.85	8.53	8.98	19.65
Y_2	10.15	8.87	17.83	38.07	14.57	8.98	10.70

Table 2. The MLEs of the parameters

Models	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{a}$	$\hat{b}$	$\hat{c}$
BGED	0.0500	0.1700	0.3640	0.012	--	--
BGLFR	0.0294	0.1701	0.3705	0.012	2.159×10^{-8}	-
BEMW	0.0298	0.1777	0.3745	0.012	2.159×10^{-8}	4.7314

Table 3. The values of $\mathcal{L}$, AIC, AICC and BIC

Models	$\mathcal{L}$	AIC	AICC	BICB
GED	-354.885	711.770	711.870	713.507
BGLFR	-270.719	547.439	548.070	552.652
BEMW	-262.407	532.814	533.895	539.765

$$v_i \pm z_{\frac{\delta}{2}}\sqrt{S_{ii}}, i = 1,2,3, c \pm z_{\frac{\delta}{2}}\sqrt{S_{44}} \quad (49)$$

where, $z_{\delta/2}$ is the upper $(\delta/2)th$ percentile of the standard normal distribution.

Data Analysis

The following data represent the American Football. For more details Csorgo and Welsh (1989). The data (scoring times in minutes and seconds) are represented in the following Table 1.

The MLEs of the unknown parameters of $BGED(\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{a})$ Kundu and Gupta (2009), $BGLFRD(\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{a}, \hat{b})$, Sarhan *et al.* (2011) and $BEMW(\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{a}, \hat{b}, \hat{c})$ models are presented in Table 2.

Table 3 contains the values of $\mathcal{L}$, AIC, AICC and BIC for the three models.

Based on the Table 3, the BEMW model fits the data better than the BGED and BGLFR model. Also, by substituting the MLEs of unknown parameters in Equation (48), the variance covariance matrix can be estimated as follows:

$$\begin{pmatrix} 0.00043193 & 0.00000213 & -0.00015334 & 0.00001292 \\ 0.00000213 & 0.00195113 & -0.00004533 & 0.00017073 \\ -0.00015334 & -0.00004533 & 0.00368595 & 0.00013815 \\ 0.00001292 & 0.00017073 & 0.00013815 & 0.01346693 \end{pmatrix}.$$

The approximate 95% confidence interval of v_1, v_2, v_3 and c are [0, 0.070535], [0.091124, 0.264276], [0.255504, 0.493496] and [4.503948, 4.958852], respectively.

Conclusion

In this study, the BEMWD whose marginals are EMW distributions is introduced. We derive some statistical and reliability measures of the new bivariate distribution. The reliability estimation for the stress-strength model is obtained. Maximum likelihood estimates are discussed. Moreover, the observed variance covariance matrix is derived. The real data set is analyzed. Finally, we conclude that, the new bivariate model fits the given real data best than the BGED and BGLFR models.

Acknowledgement

The authors are grateful to the Editor and referees for a careful checking of the details and for helpful comments that improved this paper.

Author's Contributions

A. Mustafa and M. Mahmoud: with the consultation of each other carried out this work and drafted the manuscript together. Both authors read and approved the final manuscript.

Ethics

The author declares that there is no conflict of interests regarding the publication of this article.

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