

A Review of the Recent Advances Made in the Black-Scholes Models and Respective Solutions Methods

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Abstract: The Black-Scholes model has been a major advance in finance over a period of time; this paper examines this model in some detail, in terms of the latest developments in both analytical and numerical solutions. The paper initially briefly examines historical aspects but quickly moves to a critical analysis of the works in the applications, solutions and implications for the efficiency and use of these. More specifically, this paper critically reviews the existing literature on the proposed exact as well as the numerical solutions to the Black-Scholes model. For this purpose, the exact solution literature on the Black-Scholes model includes speed based solutions and techniques based on valuation issues, time varying instruments and stochastic volatility. Similarly, the key numerical solutions include finite difference methods, the semi-discretization technique, Crank-Nicolson and the R3C scheme, the cubic spline wavelets and multi-wavelet bases method, the two-step backward differentiation formula in the temporal discretization and a High-Order Difference approximation with Identity Expansion (HODIE) scheme and fractional Black-Scholes model (TFBSM) along with Fourier analysis. This analysis reveals that transaction costs, high volatility, illiquid markets and large investor preferences are the key issues of today's financial derivatives markets, especially after the Global Financial Crisis (GFC). These issues require non-linear solutions to the Black-Scholes models; therefore, Crank-Nicolson and the R3C scheme should be focused upon more by incorporating more and more real-life assumptions of current day trading.

Keywords: Black-Scholes Model, Numerical Solution, Analytical Solution, Option Pricing

Introduction

Black-Scholes (a pricing options equation) has been used over many years in financial markets particularly for various types of options including 'exotic' options. Financial markets are becoming complex but have some common issues, such as transaction costs, illiquid markets, large investors and risks from an unprotected portfolio. Such issues need to be incorporated whilst providing either an exact or a numerical solution to different types of options. Comparatively, recent studies have attempted to incorporate all of the relevant issues in attempts to provide a solution to the Black-Scholes model. This study briefly explains all of these developments in the Black-Scholes equation along with a historical perspective of the model. It also analyses the implications of the Black-Scholes model in different

fields ranging from business (Corrado and Su, 1996) to construction projects (Barton and Lawryshyn, 2011).

This is followed by a detailed examination of exact solutions, with given additional conditions involved in the Black-Scholes questions. The introduction of nonlinearity in the Black-Scholes means that the solutions will require the use of numerical methods. Some key studies regarding the exact solutions and numerical methods are critically examined. Consequently, the first aim of this study is to explore briefly the historical perspective, implication and advancement of Black-Scholes in terms of exact and numerical solutions. The other aim is to provide a categorical analysis and a critical review of the Black-Scholes in terms of both exact and numerical solutions. A brief discussion and a summary followed by the main findings will conclude this paper.

Background of the Black-Scholes Model

It is noted that the Black-Scholes equations are still used a lot in the financial and investment world. This is because the derivative markets have become significantly important and continuing to grow. Therefore, the key concern for all finance professionals is to understand the mechanism and functioning of these markets. The proper understanding of fair prices of these derivatives can help mitigate the risks for financial professionals. The second key motivation to properly understand this mechanism is to avoid a crisis similar to the recent global financial crises (2007-2009) since it were the derivatives created for the residential mortgage in the USA that played a significant role in these crises (Hull *et al.*, 2013).

The history of options markets goes back to the middle ages, when the futures were created in order to meet the need of merchants and farmers. Consider the position of a farmer in March who will harvest in June. Now, he is uncertain about the price of grains. The option market was developed to manage these kinds of risks. Later on, these trades were formalized through the trading boards, when the Chicago Board of Trade was established in 1848 in order to bring the merchants and farmers onto a single platform. It was not long after that this board opened the Chicago Board Option Exchange in 1973 (Hull *et al.*, 2013). It was then that academicians and financial researchers started to focus on the valuation methods, such as the Black-Scholes equation to model the prices of the options.

Despite some estimation issues, as pointed out in Duan (1999; Yang, 2006; Harun and Hafizah, 2015), the Black-Scholes model has received considerable attention over past two decades-especially in underlying probability attributes of a European call option when written on a non-dividend stock. The Black-Scholes model estimates the probability of a European call option, which is frequently used in the investment decisions (see the work of Fischer Black, Myron Scholes and Robert Merton; they started publishing work related to the pricing of mainly European stock options). The focus was not on American options since the European one was much easier to deal with at the time.

More specifically, Fischer was working on the stock warrant valuation models when Scholes became interested in his work and joined his research. Based on this, they developed their initial versions of the Black-Scholes equation. The initial version of Black and Scholes (Black and Scholes, 1973) was rejected twice, by: (i) Review of Economics and Statistics and (ii) Journal of Political Economy. Later on, this work was further developed by Merton (1973) and interestingly, the Merton paper was accepted earlier than the Black and Scholes (Black and Scholes, 1973). It seems that, on grounds of fairness, Robert Merton asked the journal editor

to hold this publication until the acceptance and publication of Black and Scholes (Black and Scholes, 1973).

An acknowledgement of the importance of the Black-Scholes model came in 1997, when Myron Scholes and Robert Merton were awarded the Nobel Prize in 1997. Fischer Black passed away on 30th August 1995; otherwise, he would undoubtedly also have been one of the recipients of the Nobel Prize. This model has now become one of the most important applications of Ito calculus in financial engineering. In this context, Wilmott *et al.* (1995) pointed out that the Black-Scholes or Black-Scholes-Merton model is the basic building block of the financial derivatives theory. Further, this model played a vital role in the growth and success of financial engineering. The above discussion raises an important question as to how Black, Scholes and Merton made their breakthrough.

Theoretical Justification of the Breakthrough

In terms of options discussed earlier, researchers have used various assumptions in order to calculate the expected payoff of the European options (Feltham, 1968). But in most instances, it was difficult to arrive at the correct discount rate, which is the key element in the calculation of the expected payoff of the European option, as elaborated in section 12.2 of Hull *et al.* (2013). In order to resolve this lingering issue, Black and Scholes used the capital asset pricing model to determine the association between the markets' required return on the option and the required return on the stock.

This was a rather complicated issue since the underlying relationship was dependent upon: (i) The stock price and (ii) time. In this nexus, Merton's approach was different, involving a riskless portfolio of the option and underlying stock. This riskless portfolio was based on the argument that the return on the portfolio over a short period of time must be equal to the risk-free return. This approach is more general when compared to Black and Scholes' approach since it does not rely on the assumption of the capital asset pricing model in valuing stock options, namely, the Black-Scholes-Merton Model. This model helped, in the end, to turn around the guessing game of forward pricing of options into a mathematical science, which in turn further aided in the development of pricing more exotic options and many other different tools in financial engineering.

In addition to the assumptions mentioned, MacBeth and Merville (1979) stated some other key assumptions, namely, (i) the underlying stock pays no dividend, (ii) the risk-free rate remains constant over the period of an option, (iii) the market players can lend and borrow at the risk-free rate, (iv) the price of the stock one period ahead has a log-normal distribution with constant mean and variance and finally, (v) the number of shares in the stock outstanding remains the same.

Implications of the Black-Scholes Model

During the 1970s, after the publication of Black and Scholes (Black and Scholes, 1973), researchers focused on extracting some empirical evidences from the key financial markets, including insurance sector giants. For instance, MacBeth and Merville (1979) conducted a comparative analysis of the real market prices of call options with the prices predicted by Black and Scholes (Black and Scholes, 1973). These studies attracted researchers from other fields to use these models in the related price predictions. Years later, the Black-Scholes model has been widely used in different fields ranging from business (Corrado and Su, 1996) to construction projects (Barton and Lawryshyn, 2011).

During the 1980s, MacBeth and Merville (1980) applied the model to test the Cox call option valuation for the constant elasticity of variance diffusion processes against the Black-Scholes model. This study observed the movement of common stock prices in line with the constant elasticity of variance diffusion processes. These results explored a new horizon for the capital market investors to predict the prices of common stock along with some other financial instruments. After a couple of years, Chesney and Scott (1989) applied the Black-Scholes model to hedging the risks against underlying securities. During the same period, the model was used in the developing economies in order to establish an optimal educational policy. More specifically, here the model was used to quantify the trade-off between providing the generalizing training as well as the specific skill training, as can be observed from Miller (1990).

From 1990 to 2017 this model has been applied to many cases and in different fields ranging from construction projects (Barton and Lawryshyn, 2011) to evaluation of information technology projects (Benaroch, 2002). In the construction industry projects, Barton and Lawryshyn (2011) priced the real options under the risk-neutral measure with a closed-form solution in order to observe the association between cash flow and value of the project. This study suggested that some of the risk can be mitigated by a delta-hedging strategy, where the project is related to the market.

The Black-Scholes model was also important in information technology. Benaroch (2002) presented an approach for managing information technology investment risk through the use of options in order to optimally control the balance between reward and risk. This study extended the application of model to the establishment of an internet sales channel and their related investment. Del Giudice *et al.* (2016) conducted a review of Black-Scholes. An overview of this qualitative approach-based study reveals that most of the application of the Black-Scholes model lies in the business studies sector focusing on the financial markets, including investment in research and development-especially in pharmaceutical companies (McGrath, 1997; 2004), customer relationship

management (Maklan *et al.*, 2005), assessment of bonds and derivatives (Singh, 2014) and management and evaluation of intangible assets (Park *et al.*, 2012).

Along similar lines, Cutland *et al.* (1995) explored new horizons of implication of the Black-Scholes model, including ascertaining the value for the deposit insurance, loans for students, farm prices support, patents, pollution rights, policies for governments and drilling rights. Despite all this, the key relevant literature focuses on the stock markets in both equity and fixed income instruments (Geske *et al.*, 1983; Rubinstein, 1983; Scott, 1987).

Comparatively recently, McKenzie, Gerace and Subedar (2007) tested the underlying probabilities attributed within the model in the Australian Stock Market. Applying qualitative regression and a maximum likelihood approach, the results of this study are in line with the Black-Scholes model. This study also includes alternative approaches such as jump-diffusion and implied volatility. Hong (2004) used the data from 1994 to 2003 from Malaysian stock markets and reported that the overall Black-Scholes model prices were significantly below the market prices; both market prices and the Black-Scholes model deviated in a certain systematic pattern. The Black-Scholes model can be applied as an investment strategy in the Malaysian stock market once the systematic pattern of deviation is clear for the specific investment. On similar lines, Mohanti and Priyan (2014) applied the Black-Scholes model and dynamic hedging strategy using daily closing prices of S&P CNX Nifty index options contracts from 1 April 2008 to 31 March 2012 in the Indian stock market. This study reported that the Indian index option market is efficient.

This brief review shows that Black-Scholes model has a number of different applications. In some cases, the results are in line with the model predictions, while in other cases; it seems there are some discrepancies. Nevertheless, the model has clearly the ability to examine various types of valuation of derivatives in different markets. Second, we note that financial derivatives are relatively less used in the electricity market and indeed in the Australian power market. This paper aims to provide a critical analysis of the advancement in the exact and numerical solutions to the Black-Scholes. Such an analysis of the latest advancements will allow researchers to choose more advanced analytical and technical methods of solutions for their research studies and model solutions.

The Black-Scholes Analytical Solution

During the last couple of years, different studies have provided solutions covering different quantitative aspects of the model. For instance, Shin and Kim (2016) focused on the Black-Scholes terminal value problem and provided its solution through the Laplace transform. This study claimed that the proposed method is simpler than the existing methods. In the early 1990s, Harper

(1994) applied the generalization technique in order to provide an exact solution for the Black-Scholes equation by reducing parabolic partial differential equations to canonical form. In this generalized form, the variables corresponding to the time appear to run backwards. These fluid mechanics can be applied in other financial derivative products where there are similar information-theoretic reasons behind the products.

Later, Forsyth *et al.* (1999) applied the finite element method under stochastic volatility to provide the exact solution of the Black-Scholes equation using the money-vanilla European option. In this approach, the outgoing waves are also correctly modelled whilst the boundary equations are discretised. For future research, this proposed technique can be applied on American options since the European option was used for the same time remaining until the expiry time.

In the early 2000s, the advanced financial engineering techniques become a part of financial products issued for the housing market. During this time, Jódar *et al.* (2005) considered the final-value problem and applied the Mellin transform to provide a new method of direct solution of Black-Scholes equation. This proposed method can be applied to the relevant option pricing models.

A year later, Rodrigo and Mamon (2006) incorporated the time varying factor into different aspects of options in order to provide the exact solution of an explicit formula for the price of an option on a dividend-paying equity. In this study, Rodrigo and Mamon (2006) used options of (i) dividend-paying equity and (ii) non-dividend paying equity. As output, this provides a simple derivation of the explicit formula of an option in time dependent parameters of the Black-Scholes partial differential equation. Thus, the pricing of other return-based equity instruments is possible through this mechanism.

Bohner and Zheng (2009) went further and applied the Adomian approximate decomposition technique to provide an exact solution to the equation. The authors suggested that the proposed technique be applied when studying some other problems in finance theory.

Comparatively recently, Edeki *et al.* (2015) modified the classical Differential Transformation Method (DTM) and used its modified version, Projected Differential Transformation Method (PDTM), to provide a faster exact solution to the Black-Scholes equation for European option valuation. A critical analysis reveals that this fast solution is efficient, reliable and better than the classical DTM. Additionally, it is recommended that both linear and nonlinear stochastic differential equations be encountered in the field of financial mathematics. For future research, this algorithm can be applied on the European put options. Table 1 summarizes the above discussion of the analytical techniques.

Numerical Solutions to Black-Scholes

Forsyth *et al.* (1999) used the finite element approach to the pricing of discrete lookbacks with stochastic

volatility. Along similar lines, Tangman *et al.* (2008) considered High-Order Compact (HOC) schemes for quasilinear parabolic partial differential equations to discretise the Black-Scholes PDE for the numerical pricing of European and American options. Dremkova and Ehrhardt (2011) then presented compact finite difference schemes to solve nonlinear Black-Scholes equations for American options with a non-linear volatility function. Since a compact scheme cannot be applied on the American type options, the study used a fixed-domain transformation. Around the same time, Song and Wang (2013) applied symbolic calculation software to provide a numerical solution using the implicit scheme of the finite difference method. This study combined the time-fractional Black-Scholes equation with the conditions satisfied by the standard put options. Two years later, Uddin *et al.* (2015) presented the numerical result of semi-discrete and full-discrete schemes for European call option and put option by Finite Difference Method and Finite Element Method. In a recent study, Zhang *et al.* (2016) used the Tempered fractional derivative to price a European double-knock-out barrier option. This study analysed characteristics of the three fractional Black-Scholes models through comparison with the classical Black-Scholes model.

Much earlier, Cortés *et al.* (2005) incorporated an important aspect of errors into the numerical solution. This study applied the Mellin transformation and proposed that the errors of composite Simpson's rule or Euler's method can be avoided whilst pricing the Black-Scholes equation in real-world financial derivatives. Company *et al.* (2006) also applied Mellin transform and a delta-defining sequence of the involved generalized Dirac delta function to provide a numerical solution of the modified Black-Scholes equation. Other studies on numerical solution literature considered different aspects. For instance, Company *et al.* (2008) applied the Semi-discretization technique to deal with the issues arising as a result of a nonlinear case of interest modelling option pricing with transaction costs.

The nonlinear Black-Scholes models are now gaining popularity because most of the realistic assumptions including transaction costs, high volatility, illiquid markets and large investor's preferences can also be included. Ankudinova and Ehrhardt (2008) analysed that the Crank-Nicolson and the R3C scheme are the most accurate techniques to price the European call option. This study incorporated different volatility problems in stock price, option price and its derivatives.

Recently, the two-dimensional Black-Scholes equation was explored by Černá *et al.* (2016) using cubic spline wavelets and multi-wavelet bases. The proposed method suggests some key advantages, including (i) high-order accuracy, (ii) a small number of degrees of freedom and (iii) a relatively small number of iterations.

Table 1: Summary of analytical solutions

Study	Methodology/ Technique	Sample/Options /Markets	Contex	Conclusion/Findings	Application /Comments
Harper (1994)	Generalization	In this generalized form, the variables corresponding to the time appears to run backwards.	In order to provide an exact solution of the Black-Scholes equation, this study reduces parabolic partial differential equations to canonical form.	This study provides the exact solution of the Black- Scholes equation, elaborating the information- theoretic reasons where the time-variant variables of the Black- Scholes equation run backward.	This application from fluid mechanics can be applied in other financial derivative products where there are similar information -theoretic reasons behind the products.
Forsyth <i>et al.</i> (1999)	The finite element method is applied to provide this exact solution of the Black-Scholes equation	The money-vanilla European option is used for the analysis with the same time remaining until expiry.	This approach provides an exact solution of the Black-Scholes model considering the options under stochastic volatility.	Applying the finite element method, the limits where the Black-Scholes equation becomes first-order hyperbolic are handled correctly. Further, the outgoing waves are also correctly modelled whilst the boundary equations are discretised.	This is a good proposed technique to be applied on American options since the European option was used for the same time remaining until the expiry time.
Jódar <i>et al.</i> (2005)	Mellin transform	This study does not require any variable to provide an exact solution to the equation.	This study aims to provide a new method of direct solution using Mellin transform.	Considering the final-value problem, this study provides an exact solution of the Black -Scholes model.	This proposed method can be applied to the relevant option pricing models.
Rodrigo and Mamon (2006)	Time-varying parameters	Options of dividend-paying equity as well as the non-dividend-paying equity	This study incorporates the time-varying factor in different aspects of options in order to provide the exact formula the price of an option on a dividend-paying equity.	This study provides a simple derivation of the explicit formula of an option in time-dependent parameters of the Black-Scholes partial differential equation.	The pricing of other return-based equity instruments is possible through applying this proposed technique.
Bohner and Zheng (2009)	The Adomian approximate decomposition technique is applied in this study.	This analytical solution does not require any variable to provide an exact solution to the Equation.	The objective of this study is to present a theoretical analysis of the Black-Scholes equation in a given terminal condition.	This study provides the analytical solution of the using Black-Scholes equation the Adomian approximate decomposition technique.	The proposed echnique is suggested for application in studying some other problems in finance theory.
Edeki <i>et al.</i> (2015)	The Projected Differential Transformation Method (PDTM), a modification of classical Differential Transformation Method (DTM)	Applied on the Black-Scholes equation for European option valuation	This study aims to apply PDTM on the Black-Scholes equation for European option valuation for a faster exact solution.	The results of this study indicate that the proposed method is efficient, reliable and better than the classical DTM. Furthermore, it is recommended that both linear and nonlinear stochastic differential equations be encountered in the field of financial mathematics.	The same algorithm is expected to be applied on the European put options.
Shin and Kim (2016)	Laplace transform	In order to provide the solution to the Black-Scholes terminal value problem, this study does not require any variable.	The aim of this study is to check the solution of the Black-Scholes terminal value problem since this problem is closely related to the put and call options of the stock marke	Applying a proposed simpler method than the existing, Laplace transform, this study provides the exact solution to the terminal value problem of Black-Scholes.	Looking at the application of the Euler-Cauchy equation, this proposed method can be applied to model any other derivative products.

Rao (2016) applied the two-step backward differentiation formula in the temporal discretization and a High-Order Difference approximation with Identity Expansion (HODIE) scheme-concerned with the generalized Black-Scholes models for European call option. This study presents the case of the European call option. The solution noted second-order accuracy in time and third-order accuracy in space.

In a recent study, Zhang *et al.* (2016; Yang, 2006) applied fractional Black-Scholes model (TFBSM)

along with Fourier analysis; this study extends different numerical solution literature on the price change of the underlying fractal transmission system. First, this study derives the fractional Black-Scholes model with an α -order time fractional derivative by applying numerical simulation. Second, the proposed techniques and method are used to price different European options. The review of the above discussion on the numerical solution is categorized and presented chronologically in Table 2.

Table 2: Numerical solutions to Black-Scholes

Study	Methodology/ Technique	Sample/Options /Markets	Context	Conclusion/Findings	Application /Comments
Cortés <i>et al.</i> (2005)	Mellin transform	Black-Scholes for pricing stock options	Along with providing numerical solutions, this research also focuses on incorporating the errors of previous proposed numerical solutions.	Considering the final value problem, this paper provides a new method to solve the Black-Scholes equation.	The errors of composite Simpson's rule or Euler's method can be avoided whilst pricing the Black-Scholes equation in real-world financial Derivatives.
Company <i>et al.</i> (2006)	This study applied two key techniques: (i) the delta-defining sequence of the involved generalized Dirac delta function and (ii) the Mellin transform.	Stock options with discrete dividend payments	This study attempts to provide a numerical solution of the modified Black-Scholes equation in order to model the stock option valuation with discrete dividend payments.	This study obtained an integral formula for the required numerical solution.	In addition to a numerical solution, this study also provides the numerical quadrature approximations along with some illustrative examples.
Company <i>et al.</i> (2008)	Semi-discretization technique	Focusing on the options with high transaction costs like vanilla call option	The objective of this study was to deal with the issues arising as a result of nonlinear case of interest modelling option pricing with transaction costs.	This study provides the numerical solution of the Black-Scholes option pricing partial differential equations applying the semi-discretization technique.	The proposed method is applicable on other forms, such as the standard put option.
Tangman <i>et al.</i> (2008)	Fourth-Order Compact Finite Difference (FOCFD)	European and American options, where the results validate the accurate numerical solution for European options only	This study applies the High-Order Compact (HOC) schemes to discretise Black-Scholes PDE in order to get numerical pricing of European as well as American options. For this purpose, quasilinear parabolic partial differential equations are used.	This study concludes that a HOC scheme with grid stretching (in association with asset price dimension) provides an accurate numerical solution in case of European options under stochastic volatility.	Considering the results of this study, coarse meshes can be used to obtain high accuracy in case of high-frequency data in option markets such as the American and European options.
Ankudinova and Ehrhardt (2008)	Crank-Nicolson and the R3C scheme are the most accurate techniques to price the European call option.	Both of the options including American and European with a volatility considering different facts: (i) Stock price, (ii) time, (iii) option price and (iv) derivatives	As a result of some realistic assumptions, including transaction costs, high volatility, illiquid markets and large investor's preferences; nonlinear Black-Scholes models are gaining popularity	This study presents different numerical discretization schemes for European options for various volatility models consisting of (i) the Leland model, (ii) the Barles and Soner model and (iii) the Risk adjusted pricing methodology mode.	Following the results of this study, the generalized forms of Numerov-type and Crandall-Douglas scheme can be applied for the numerical solution of nonlinear Black-Scholes equations.
Dremkova and Ehrhardt (2011)	This study applies FOCFD using a fixed-domain Transformation proposed by Sevčovic.	Standard options generally known as plain vanilla options of American type since a closed-form solution to non-linear Black-Scholes equation does not exist; resultantly, solved numerically	This study addresses some of the issues in classical Black-Scholes model assumptions as a result of (i) transaction costs, (ii) illiquid markets, (iii) large investors and (iv) risks from an unprotected portfolio. As a result of these issues, the model faces nonlinear, possibly degenerate, parabolic diffusion issues.	This study indicates that compact methods are the better option for efficient modelling of American options applying the fixed-domain transformation technique.	These results motivate the implementation of the discrete artificial boundary condition in this framework.
Song and Wang (2013)	Finite difference method	European and American put options	Based on the time-fractional Black-Scholes equation; the aim of this study is to provide the numerical solution to the standard put options using symbolic calculation software.	The successful numerical solution provided in this study indicates that the finite difference method in this framework is effective with less computational work in order to solve the fractional partial differential equation.	This fractional model can be applied to model the price of other financial derivatives, including swaps, warrants and so on.

Table 2. Continue

Uddin <i>et al.</i> (2015)	This study applies the Finite Element Method (FEM) followed by the weighted average method, where different weights are used for numerical approximations.	European call and put options	This study attempts to solve European call and put options and apply different numerical and analytical methods.	This study presents numerical results of the semi as well as full discrete schemes by (i) the Finite Difference Method and (ii) the Finite Element Method in order to model the European call and put option pricing.	The linear algebra solvers can be applied to different forms of options to verify the solvers.
Černá <i>et al.</i> (2016)	Cubic spline wavelets and multi-wavelet bases	For all types of options	The objective of this paper is to provide numerical solutions for the two-dimensional Black-Scholes equation.	By providing a numerical solution through cubic spline wavelets, this study concludes that this proposed method is suggested due to some key advantages, including (i) high-order accuracy, (ii) a small number of degrees of freedom and (iii) a relatively small number of iterations.	This method can be applied to a wide range financial derivatives for the computation of their market value at a given time.
Zhang <i>et al.</i> (2016)	Tempered fractional derivative and numerical simulation; additionally, a Fast Bi-Conjugate Gradient stabilized method (FBi-CGSTAB) is used for two key purposes-to reduce (i) storage space and (ii) computational cost per iteration.	This technique is applied to price a European double-knock-out barrier option. Further, a comparative analysis is conducted on the characteristics of the three fractional Black-Scholes models and classical Black-Scholes model.	Large movements and jumps over the small time steps are the important characteristics of the dynamic process of stock price changes over the last couple of years. In order to capture these dynamics, the Finite Moment Log Stable (FMLS), KoBoL and CGMY models are considered the most popular modelling framework. This study provides a numerical solution of these models.	The findings of experimental data indicate that experimental results are in line with the theoretical analysis.	Based on these results, the numerical techniques can be applied in related fractional models for pricing different options.
Rao (2016)	The numerical method used in this study is based on the two-step backward differentiation formula in the temporal discretization and a High-Order Difference approximation with Identity Expansion (HODIE) scheme.	European call option	This numerical solution is more concerned with the generalized Black-Scholes models for the European call option.	This study presents a high-order numerical method in order to provide the numerical solution of the generalized Black-Scholes models especially for the case of the European call option. Additionally, this solution gives second-order accuracy in time and third-order accuracy in space.	Since these numerical experiments are conducted in order to support theoretical aspects of the financial derivatives, these numerical solutions can be extended to the European put option.
Zhang <i>et al.</i> (2016)	The numerical simulation of this Time Fractional Black-Scholes Model (TFBSM) along with Fourier analysis	A special case of European options	This research extends different numerical solution literature on the price change of the underlying fractal transmission system.	This research is two-fold. First, it derives the fractional Black-Scholes model with an α -order time fractional derivative applying numerical simulation. Second, the proposed techniques are used to price different European options.	These numerical techniques are expected to apply in other similar fractional models to price European options.

Discussion of Analytical Solutions

The exact solution literature provided in Table 1 addresses a couple of real-world phenomena of the financial markets. Considering these phenomena, this literature can be categorized into the following:

- i. A speed-based solution in which Edeki *et al.* (2015) applied Projected Differential Transformation Method (PDTM), a modification of classical Differential Transformation Method (DTM) for a faster solution to Black-Scholes equation:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (k-1) \frac{\partial v}{\partial x} - kvV(x, h+1) \\ &= \frac{1}{h+1} \left[\frac{\partial^2 V(x, h)}{\partial x^2} + (k-1) \frac{\partial V(x, h)}{\partial x} - kV(x, h) \right] \end{aligned} \quad (1)$$

- ii. Valuation issues-based techniques using which Shin and Kim (2016) addressed the terminal value problem of the Black-Scholes model by using the Laplace transformation:

$$\begin{aligned} u^2 \mathfrak{f}(v) - uv(x, 0) - v_i(x, 0) \\ + \frac{1}{2} \sigma^2 u s^2 \mathfrak{f}(v_{ss}, t) + ru \mathfrak{f}(v_{ss}, t) - r \mathfrak{f}(v) = 0 \end{aligned} \quad (2)$$

Jódar *et al.* (2005) applied the Mellin transformation to solve the final-value problem:

$$\begin{aligned} \frac{\partial C}{\partial t}(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S, t) + rS \frac{\partial C}{\partial t}(S, t) - rC(S, t) \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\begin{aligned} &-p(\alpha + i\tau) + \\ &\frac{\sigma^2}{2} S^2 (\alpha + i\tau)(\alpha + 1 + i\tau) \\ &S^{-2} - rS(\alpha + i\tau)S^{-1} - r \end{aligned} \right] \\ \times f^*(\alpha + i\tau) e^{-p(\alpha + i\tau)(\tau - T)} d\tau = 0 \end{aligned} \quad (3)$$

- iii. A time-varying instruments-based technique that Rodrigo and Mamon (2006) used time-varying parameters for the options of dividend paying equity as well as the non-dividend paying equity:

$$\begin{aligned} V(s, t) &= \frac{K}{K^-} \exp \left[- \int^t \left(r(u) - \frac{r_c}{\sigma_c^2} \sigma(u) du \right) \right] V^-(s^-, t^-), \\ s^- &= \frac{K^-}{K} \exp \left[- \int^t \frac{r_c}{\sigma_c^2} \sigma(u) + D(u) - r(u) du \right] s, \\ t^- &= - \frac{1}{\sigma_c^2} \int^t \sigma(u)^2 du + T^- \end{aligned} \quad (4)$$

- iv. A stochastic volatility-based technique with which Forsyth *et al.* (1999) applied the finite element method to provide this exact solution of the Black-Scholes equation:

$$\begin{aligned} U_t = \frac{v\alpha^2}{2} U_{\alpha\alpha} + \rho\sigma v\alpha U_{\alpha v} + \frac{\sigma^2 v}{2} U_{vv} \\ + r\alpha U_{\alpha} + (K(\theta - v) - \lambda v) U_v - rU \end{aligned} \quad (5)$$

All of these proposed techniques provide analytical and/or quasi-exact solutions by incorporating different issues of financial derivatives. Therefore, these techniques have their importance for the applicability purpose.

Discussion on Numeric Solutions

The nature of the conditions used in the valuation methods make models such as Black-Scholes a non-linear one and they then require numerical approaches to reach satisfactory solutions. Some of the aspects that influence the nature of the model are: Transaction costs, illiquid markets, large investors, risks from an unprotected portfolio, weights allocation methods along with the long movements and jumps over the small time steps are the common problem in the financial derivatives markets. All of these issues are incorporated into different studies (Dremkova and Ehrhardt, 2011; Song and Wang, 2013; Tangman *et al.*, 2008; Uddin *et al.*, 2015; Zhang *et al.*, 2016), all of which attempted to provide the numerical solution to the Black-Scholes model using finite difference methods. Other proposed techniques in numerical solution approaches are as follows:

- i. Semi-discretization technique (Company *et al.*, 2008):

$$\begin{aligned} U_{\tau} - \frac{S^2}{2} \sigma^2 U_{SS} - rSU_S + rU = 0 \quad 0 < S < \infty, 0 < \tau \leq T \\ \tau = T - t \\ U(S, 0) = \max(0, S - E) \\ U(0, \tau) = 0, \lim_{S \rightarrow \infty} \frac{U(S, \tau)}{S - Ee^{-r\tau}} = 1 \end{aligned} \quad (6)$$

- ii. Crank-Nicolson and the R3C scheme-the most accurate techniques to provide a solution to the nonlinear Black-Scholes models (Ankudinova and Ehrhardt, 2008):

$$\begin{aligned} D_k^+ U_i^n + D_k^- U_i^{n+1} \\ = s_i^n (D_h^2 U_i^n + D_h^0 U_i^n) + s_i^n (D_h^2 U_i^{n+1} + D_h^0 U_i^{n+1}) \\ + (1 + D) (D_h^0 U_i^n + D_h^0 U_i^{n+1}) + D_h^2 U_i^n + D_h^2 U_i^{n+1} \\ D_k^+ U_i^n = (1 + s_i^n) \left(\left(\frac{1}{2} + A_1 \right) D_h^2 U_i^n + \left(\frac{1}{2} + B_1 \right) D_h^0 U_i^n \right) \\ + (1 + s_i^n) \left(\left(\frac{1}{2} + A_2 \right) D_h^2 U_i^{n+1} + \left(\frac{1}{2} + B_2 \right) D_h^0 U_i^{n+1} \right) \\ + D \left(\frac{1}{2} + B_1 \right) D_h^0 U_i^n + D \left(\frac{1}{2} + B_2 \right) U_i^{n+1} \end{aligned} \quad (7)$$

- iii. The cubic spline wavelets and multi-wavelet bases method to provide a solution for the two-dimensional Black-Scholes equation (Černá *et al.*, 2016):

$$\begin{aligned} \mathcal{L}_{BS}(V) = \frac{\sigma_1^2 S_1^2}{2} \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \\ + \frac{\sigma_2^2 S_2^2}{2} \frac{\partial^2 V}{\partial S_2^2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV \\ \frac{(V_{i+1}^- \cdot v)}{\tau} - \theta a(V_{i+1}^-, v) - (1 - \theta) a(V_i^-, v) \\ = \frac{(V_i^- \cdot v)}{\tau} + \theta (f_{i+1}, v) + (1 - \theta) (f_i, v) \\ \text{Where } v \in H_0^1(\Omega), a(u, v) = (\mathcal{L}_{BS}(u), v) \end{aligned} \quad (8)$$

- iv. A two-step backward-differentiation formula in the temporal discretization and a HODIE scheme (Rao, 2016):

$$Lu(x,t) \equiv \frac{\partial u}{\partial t} - a_2(x,t) \frac{\partial^2 u}{\partial x^2} - a_1(x,t) \frac{\partial u}{\partial x} - a_0(x,t)u = f(x,t), (x,t) \in \Omega \quad (9)$$

- v. A TFBSM along with Fourier analysis (Zhang *et al.*, 2016):

$$\begin{aligned} \frac{\partial^\alpha C(S,t)}{\partial t^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + (r-D)S \frac{\partial C(S,t)}{\partial S} - rC(S,t) &= 0 \\ (S,t) \in (B_d, B_u) \times (0,T) & \\ C(B_d,t) = p(t), C(B_u,t) = q(t) & \\ C(S,T) = v(S) & \end{aligned} \quad (10)$$

It has been noted that the transaction costs, high volatility, illiquid markets and large investor's preferences are some key issues in today's financial derivatives markets, especially after the event of GFC. These inclusions require the solving of a nonlinear solution to the Black-Scholes models. It is recommended therefore that the Crank-Nicolson and the R3C scheme be focused on more because they can accommodate more and more real life assumptions of current-day trading (Ankudinova and Ehrhardt, 2008).

Conclusion

This paper critically analyse the recent advances made in the Black-Scholes model and solution methods. The authors examine the historical aspects, various applications, analytical and numerical solutions and efficiency and use of the same. The historical perspective and implication section of this study reveals that the Black-Scholes model is able to examine various types of valuations of derivatives in different markets. Many researchers have attempted to obtain the solution of the Black-Scholes model analytically or numerically, thereby adopting and using various direct and iterative methods, respectively. In the literature on the Black-Scholes model, exact solutions come in the form of speed-based solutions and techniques based on valuation issues, time-varying instruments and stochastic volatility. All of these proposed techniques provide analytical and/or quasi-exact solutions by incorporating different issues of financial derivatives. Therefore, these techniques have advanced their importance in the applicability as well as purpose. The numerical solutions include finite-difference methods, semi-discretization technique, Crank-Nicolson method, the R3C scheme, cubic spline wavelets and multi-wavelet bases method, the two-step backward differentiation formula in the temporal discretization

and a High-Order Difference approximation with Identity Expansion (HODIE) scheme, fractional Black-Scholes model (TFBSM) using a Fourier analysis. The main findings show that transaction costs, high volatility, illiquid markets and large investor preferences are key issues of today's financial derivatives markets-especially after the Global Financial Crisis (GFC). Complexity issues require non-linear solutions to the Black-Scholes model; therefore, the Crank-Nicolson method and the R3C ought to should be focused on more by incorporating more real-life assumptions of current day trading. Moreover, the Finite difference methods are by far the simplest, except when mesh adaptively is required; in which case it is rather difficult to control the numerical error. It is noted that the FEM performs better with respect to the convergence than the explicit FDM particularly when small number of time points is used. The FEM converges to the solution when the explicit scheme of FDM diverges.

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Author's Contributions

All the authors contributed equally to writing the manuscript and they had an equal say in reviewing and approving the final version.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and there are no ethical issues involved.

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Abbreviations

DTM	=	Differential Transformation Method
FBI-CGSTAB	=	Fast Bi-Conjugate Gradient Stabilized Method
FEM	=	Finite Element Method
FMLS	=	Finite Moment Log Stable
FOCFD	=	Fourth-Order Compact Finite Difference
GFC	=	Global Financial Crisis
HOC	=	High-Order Compact
HODIE	=	High-Order Difference approximation with Identity Expansion
PDTM	=	Projected Differential Transformation Method
TFBSM	=	Fractional Black-Scholes Model